

Introductory Solid Mechanics

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Preface

The subject of mechanics of materials is perhaps the most fundamental topic in engineering science. It is among the most widely taught and studied courses for engineers. It has been exhaustively analyzed, picked apart, described, and textualized. There are dozens of texts (with titles like Mechanics of Materials, Strength of Materials, Solid Mechanics, etc.) in use at colleges and universities around the world, some with decades-long histories and more than 10 editions. The logical question at this point may be “Why another?”

The motivation is two-fold. First, higher education in the United States has become prohibitively expensive. As of this publication, many textbooks cost in excess of \$300. Some of the most notable contain entire chapters of extra information, hundreds of pages of homework problems, and reams of appendices that the average student will never use. There is certainly value in all of that, and those texts serve as excellent references if students or graduates need an odd advanced topic, but the vast majority of students who have to buy those books do not recoup the investment. This text is written explicitly to cover a single-semester introductory course. While there are discussions of some advanced materials, it is by no means exhaustive. Relevant material is included, but extraneous “bonus” topics, problems, and references are excluded in the interest of accessibility.

The second, and perhaps more important, reason for writing this book is to overcome what could be seen as shortcomings in most current books. As stated previously, the subject is one of the oldest and most developed in engineering curricula. In the past, it was critical that engineers be able to simplify complex problems into manageable forms, know how to manipulate various equations and models, and identify as many surrogates and simplifications as possible for as wide a variety of situations as possible. In the modern era with the advent of computing technology, previously untenable problems are now solved as a matter of rote. The focus of what engineering students need to learn has shifted; gone is the value of dozens of convenient work-arounds and simplifications, replaced by a pressing need for understanding the basic physics surrounding deformable body mechanics. That is not to say that knowledge of those models is no longer important. Rather, the focus has shifted from *how* to appropriately use a model to *when* and *why* a model might be appropriate.

Each model presented in the text (bars, shafts, beams, plane stress) is derived from first principles. The focus of the derivation is on choice and application of

assumptions. Why are certain assumptions made and what is the consequence of their inclusion or exclusion? When is the resulting model valid? Under what circumstances are the solutions of each model a reasonable representation of reality? Students will learn how to use the models for direct calculation, but will also learn when each might be a valid simplification for use in computational solutions such as FEA. Examples throughout the text are solved only after explicitly determining that the problem is appropriate (i.e., satisfies all assumptions and requirements) for the given model.

The text is intended to support a single-semester introductory course. It is organized in exactly the order the author teaches his own class, and almost all material (cover-to-cover) is taught. There are advanced topics, denoted by an asterisk in the section title, which may be considered optional. Each specific topic is treated in such a way as to prepare students for more advanced material; after completing this text, students will be ready for courses in applied mechanics, design, energy methods, elasticity, finite element analysis, or continuum mechanics.

Chapter 1

Mechanics of Materials

Mechanics of materials (more generally known as **solid mechanics**) is one of the most fundamental areas of engineering science. The topics covered in this subject are incredibly important and ultimately useful to many applications. The obvious starting point for an investigation of the subject is to ask, “What is solid mechanics?” To develop a satisfactory answer to this question (which will be attempted over the course of this text), we need to start by stepping back and looking at a bigger picture.

The engineering discipline may be broadly divided into two overarching categories, **Engineering Science** and **Design**. Design generally refers to the application of analysis, engineering skills, and old-fashioned creativity to produce a final product. This product may be an entirely new object, or it could be something as simple as an updated maintenance schedule. Engineering science is the set of fundamental physical principles, laws, and relations that are used to guide the design process and make educated decisions. These principles may draw from physics, chemistry, biology, or even social sciences. In any case, the basic science is developed with a clear goal in mind: application. In most cases, engineering science is divided into two primary subjects, **mechanics** and **thermodynamics**. Most specialties fall under the purview of one of these two. Thermodynamics is the study of energy and how it is transferred. Mechanics is the study of motion. If we dig deep enough, the line between these two subjects becomes indistinct, but for our purposes it suffices to consider them separately.

Now, when we say mechanics of materials or solid mechanics, we understand that we are studying the motion of solid material. There is a further implication in the title, however; it is not enough that an object be moving. A pinewood derby car rolling down a ramp may be an obvious case of mechanics, but it is generally considered irrelevant with regards to the specific subject of solid mechanics. The motions we are interested in are *internal* to the solid body.

1.1 Statics

At this point we have all had some exposure to the field of mechanics. Perhaps the most basic representation of this is in the case of a static problem, in which the state of motion is that there is none. A brief review of this topic follows.

1.1.1 Vector Algebra

An important tool in the analysis of mechanics problems is the **vector**, an object with magnitude and orientation. The magnitude of a vector is its length, and the orientation may be measured as an angle from some datum. Perhaps a more useful means of expressing vectors is in terms of a standard set. Consider 3D Cartesian space; the axes of this coordinate system are vectors themselves. The axes form a **basis** of the space, which means that any other vector in this space can be expressed as a linear combination of the axes. That is, any vector may be represented in terms of the contributions of each axial component.

Let us express our base vectors as:

$$\hat{i} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \hat{j} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad \hat{k} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad (1.1)$$

These base vectors (known as the **Cartesian base vectors**) are unit vectors in the x -, y -, and z -directions. A vector \vec{a} that extends two units in the x -direction and two units in the y -direction is constructed from the basis vectors (Figure 1.1).

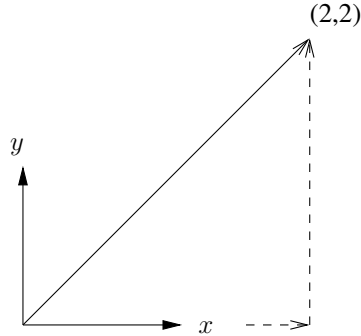


Figure 1.1: Composition of a vector from the cartesian basis

$$\vec{a} = 2\hat{i} + 2\hat{j} = \begin{Bmatrix} 2 \\ 2 \\ 0 \end{Bmatrix} \quad (1.2)$$

As shown in Figure 1.1, the vector forms the hypotenuse of a right triangle, the legs of which are the x - and y -components. The magnitude of the vector is

given by Pythagorean theorem and its angle relative to the x -axis is determined by simple trigonometry.

$$\vec{a} = 2\sqrt{2} \angle 45^\circ \quad (1.3)$$

The two expressions in (1.2) and (1.3) are equivalent. One additional point of notation: vectors are labeled using the arrow notation (as shown in equation (1.3)), using braces, or by using a boldface variable.

$$\vec{a} = \{a\} = \mathbf{a} \quad (1.4)$$

Vector labels are interchangeable, but boldface is the primary notation in this text. The component form is generally more convenient for the present work, and will be used almost exclusively. Arithmetic operations involving vectors are straightforward in this notation. Addition of two vectors requires only scalar addition of corresponding components. Multiplication of a vector by a scalar is performed by multiplying each component individually by the scalar. This is illustrated in equation (1.5), where \mathbf{a} and \mathbf{b} are vectors and k is a scalar.

$$\begin{aligned} \mathbf{a} &= \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} & \mathbf{b} &= \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \\ \mathbf{a} + \mathbf{b} &= \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} + \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \begin{Bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{Bmatrix} \\ k\mathbf{a} &= k \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} ka_1 \\ ka_2 \\ ka_3 \end{Bmatrix} \end{aligned} \quad (1.5)$$

Note that multiplication by a scalar changes the magnitude of a vector, but not the direction.

Vector multiplication is rather more complicated, and in fact is dependent on the purpose of the multiplication process. There are two primary methods of multiplying vectors. The first method, typically called the **dot product** (the symbol for this product operation is a raised dot, \cdot), is often used to find the projection of a vector in the direction of another vector, or the amount of the vector oriented in a certain direction, scaled by the second vector (Figure 1.2). Using component notation, the dot product is simply defined as the sum of the products of corresponding components. Using \mathbf{a} and \mathbf{b} from equation (1.5):

$$\mathbf{a} \cdot \mathbf{b} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \cdot \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (1.6)$$

Note that the result of this operation is a scalar. Because of this, the dot product is often called the **scalar product** or the **inner product**. Alternatively (but equivalently), the dot product can be defined using the magnitudes of the vectors and the angle between them, avoiding the need for individual components.

$$\mathbf{a} \cdot \mathbf{b} = (\|\mathbf{a}\|) (\|\mathbf{b}\|) \cos(\theta) \quad (1.7)$$

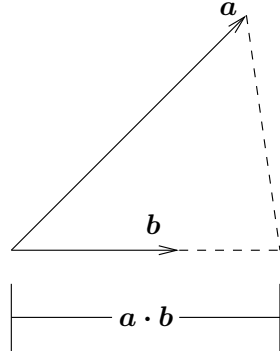


Figure 1.2: Illustration of the dot product between two vectors

To find an unscaled projection of vector \mathbf{a} in another direction, then the norm of the second vector should be 1. In other words, the dot product between a vector \mathbf{a} and a unit vector in the direction of \mathbf{b} produces the component of \mathbf{a} in that direction.

$$\mathbf{a}_b = (\|\mathbf{a}\|)(\|\mathbf{u}_b\|) \cos(\theta) = \|\mathbf{a}\| \cos(\theta) \quad (1.8)$$

If the dot product represents a projection of a vector in a direction, then the projection of a vector in its own direction should be the entire vector. Using this logic, the magnitude of a vector (also called its **norm**) can be defined as:

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2} \quad (1.9)$$

An alternative definition of vector multiplication, known as the **cross product** (denoted by a cross symbol, \times), results in a vector rather than a scalar. The cross-product is only defined for three-dimensional vectors; given two vectors in 3D space, their cross-product generates a vector that is normal to them both. The computation of the cross product is performed by taking the determinant of a matrix formed from the basis and the two vectors in question:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \times \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1) \\ &= \begin{Bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{Bmatrix} \end{aligned} \quad (1.10)$$

Note that the cross product is not commutative. Order of multiplication matters. The cross product follows the right-hand rule; a right hand curled from

\mathbf{a} into \mathbf{b} will have the thumb pointed in the direction of the cross product, as shown in Figure 1.3.

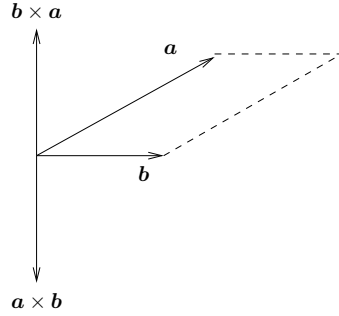


Figure 1.3: Illustration of the cross product between two vectors

1.1.2 Newton's Laws and the Equilibrium Equations

The fundamental purpose of the field of mechanics is the study of motion of a body. In order to understand motion, one must understand the forces associated with motion. Our most basic description of these relations is known as **Newton's Laws of Motion**. The two basic assumptions involved in using Newton's laws are (i) that the forces act on a single point and (ii) the observer is in an inertial reference frame (that is, the observer is not accelerating). The laws are generally expressed as follows:

1. Inertia: An object maintains its state of motion unless acted on by an unbalanced force.
2. $\mathbf{F} = m\mathbf{a}$: The force is equal to the rate of change of the momentum, or mass times acceleration if mass is constant.
3. Every action has an equal and opposite reaction.

Note that the force and the acceleration are vector quantities. For the purposes of mathematically describing motion, the second law is most directly useful. If an object is static, it is not accelerating.

$$\mathbf{F} = m(\mathbf{0}) = \mathbf{0} \quad (1.11)$$

Or in component form:

$$\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (1.12)$$

If an object is static, the net force in each direction is zero. Newton's laws, unfortunately, are not sufficient to describe the state of motion of a finite body, only a particle. Euler extended Newton's laws to a finite body by accounting

for a continuum constructed of many particles. The results of Newton's laws, i.e., that the total forces are zero if the body is static, still hold. In addition to that, however, distance effects due to the separation of individual particles must now be considered. The moment vector captures these effects, and is associated with rotational motion, or orientation. A single point has no orientation, hence moments are not necessary for the analysis of its motion. Similarly to the force constraints, the net moment must be zero for a body to be static.

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (1.13)$$

The moment vector is related to a rotation about its own axis; for example, M_x is pointed in the x -direction, and is associated with a rotation about the x -axis. Moments may be modeled as independent vectors (sometimes called **couples** or **pure moments**). Generally, the moment is caused by a force acting at a distance, defined by:

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \quad (1.14)$$

\mathbf{r} is the position vector from the point where the moment is being computed to the force \mathbf{F} . This method of computing the moment gives all information about the moment; every component and its direction (sign) is wholly determined by the cross product.

The moment vector is included in a free-body diagram using two possible notations. The first is a double-headed arrow pointing in the direction of the moment vector. Alternatively, the moment may be drawn using a curved arrow showing the direction of the rotation with which it is associated. Since the cross product produces a vector normal to the multiplied factors, the double-headed arrow will point directly out of the plane of the curved arrow; in other words, the curved arrow notation for a particular moment will “wrap around” the equivalent double headed arrow, as shown in Figure 1.4. In one-dimensional problems, any moment vectors will be oriented along the object, so a curved arrow would point into the page. Similarly, for 3D problems it can be difficult to visually judge the orientations of various curved arrows. In both of those cases, the double-headed arrow notation is preferred. However, in a two-dimensional free-body diagram, forces and distances would all act in a single plane, so a double-headed vector normal to that would point into or out of the page. For this situation, the curved arrow notation is ideal.

In many cases, especially in two dimensional problems, it is perhaps simpler to consider a scalar formulation of a moment. The magnitude of a moment is the force times the *perpendicular* distance to a point. This must be the same moment that would be computed with the cross product, so it also follows the right-hand rule. That is, moments associated with counterclockwise rotations are positive. The perpendicularity condition is simple to use in 2D; x -components of forces are multiplied by y -distances and vice versa. As previously discussed, the moment is typically shown using a curved arrow notation showing its rotational effect in the plane (Figure 1.5).

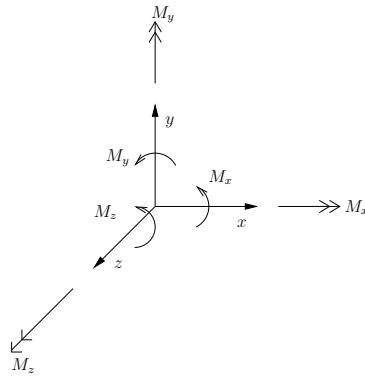


Figure 1.4: Moment vector symbols

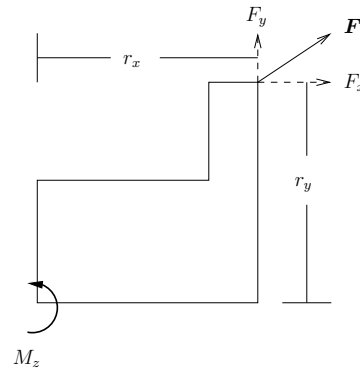


Figure 1.5: Moment vectors in two dimensions

Care is required with the signs when using the scalar moments. In general, both the distance and the force can affect the sign of a moment. Equation (1.15) is a clear, concise formula for determining the scalar moment with appropriate sign in 2D space, as long as the input quantities are defined correctly.

$$M = F_y r_x - F_x r_y \quad (1.15)$$

F_x and F_y are the x - and y -components of the force vector, and r_x and r_y are the x - and y -components of the distance vector measured *from the point where the moment is computed to the force* (the same definition as with the cross-product), as shown in Figure 1.5. In Figure 1.5, all force and distance components are positive (oriented to the right and upward).

For a finite body, the net force and moment are the vector sums of all forces and moments acting on the body. Equivalently, the equations of static

equilibrium are most often expressed as:

$$\begin{aligned}\sum F_x &= 0 & \sum M_x &= 0 \\ \sum F_y &= 0 & \sum M_y &= 0 \\ \sum F_z &= 0 & \sum M_z &= 0\end{aligned}\tag{1.16}$$

In two dimensions, only three equations are required. Most often, the two in-plane force equations and the moment equation normal to them (associated with the rotation in the plane) are used, but there may be alternatives.

Example 1.1 (3D Static Equilibrium).

PROBLEM STATEMENT:

Consider the tire iron shown in Figure 1.6(a). While attempting to change a flat tire, you attach the wrench end of the tire iron to a lugnut, and push downward on the opposite end of the iron. The lugnut is tightly locked down, and remains static. If you push down on the tire iron with 300 N force, what are the reactions at the lugnut?

KNOWNs:

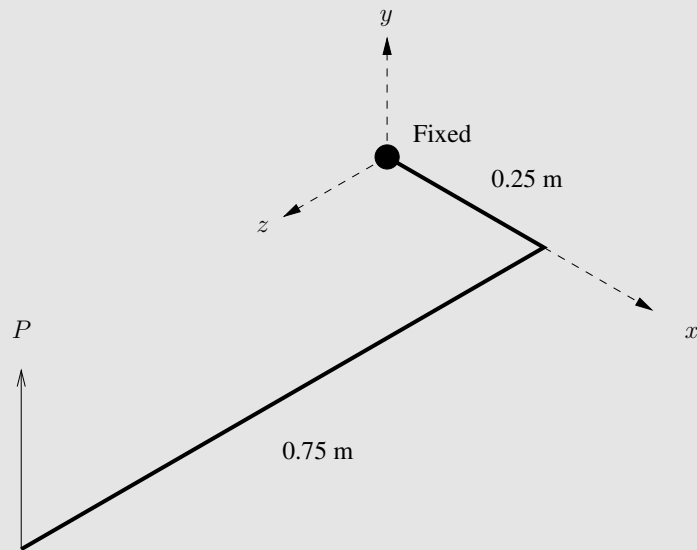
- $P_y = -300$ N
- Dimensions as shown in Figure 1.6(b)

ASSUMPTIONs:

- The tire iron experiences negligible deformation under the load
- The lugnut acts as a fixed support



(a) Tire iron



(b) Free-body diagram

Figure 1.6: Example 1.1

ANALYSIS:

The first step in any static analysis is to draw a free-body diagram, as shown in Figure 1.6(b). The origin of the coordinate system is at the lugnut. The long arm of the tire iron is parallel to the z -axis, the short arm aligns with the x -axis, and the applied force is parallel to the y -axis. Dimensions are shown in Figure 1.6(b). Since we assume the tool does not deform significantly, the distances, directions, and shapes are all the same as the unloaded state.

If the lugnut remains static, we may assume that it provides all six possible

loads; forces R_x , R_y , and R_z and moments M_x , M_y , and M_z (not shown on free-body diagram for simplicity).

Define the force and position vectors:

$$\mathbf{P} = \begin{Bmatrix} 0 \\ P_y \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -300 \text{ N} \\ 0 \end{Bmatrix} \quad (1.17)$$

$$\mathbf{r} = \begin{Bmatrix} 0.25 \text{ m} \\ 0 \\ 0.75 \text{ m} \end{Bmatrix} \quad (1.18)$$

Note that $P = -300$ because the force is downward, but the vector is drawn upward in the free-body diagram. With these vectors, determine the moment at the lugnut due to the applied force:

$$\mathbf{M}_P = \mathbf{r} \times \mathbf{P} = \begin{Bmatrix} 225 \\ 0 \\ -75 \end{Bmatrix} \text{ N}\cdot\text{m} \quad (1.19)$$

With all quantities fully-defined, apply the equilibrium conditions:

$$\begin{aligned} \sum F_x = 0 = P_x + R_x &\implies R_x = 0 \\ \sum F_y = 0 = P_y + R_y &\implies R_y = 300 \text{ N} \\ \sum F_z = 0 = P_z + R_z &\implies R_z = 0 \\ \sum M_{Ox} = 0 = M_{Px} + M_x &\implies M_x = -225 \text{ N}\cdot\text{m} \\ \sum M_{Oy} = 0 = M_{Py} + M_y &\implies M_y = 0 \text{ N}\cdot\text{m} \\ \sum M_{Oz} = 0 = M_{Pz} + M_z &\implies M_z = 75 \text{ N}\cdot\text{m} \end{aligned} \quad (1.20)$$

Take care with the notations in the moment terms; $\sum M_{Ox}$ is the sum of moments *about* the x -axis, M_{Px} is the x -component of the moment at the origin due to the force P , and M_x is the x -component of the unknown moment.

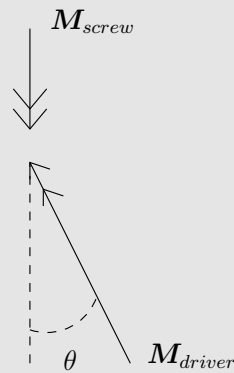
Example 1.2 (3D Static Equilibrium).

PROBLEM STATEMENT:

While attempting to change the doorknob at your new home, you notice that the current knob is shaped such that a screwdriver meets the screw at a 30° angle, as shown in Figure 1.7(a). If the screw requires a moment of 10 N·m about its axis to break free, what is the moment required on the screwdriver?



(a) Doorknob with screwdriver



(b) Free-body diagram

Figure 1.7: Example 1.2

KNOWNs:

- $M_{screw} = 10 \text{ N}\cdot\text{m}$
- $\theta = 30^\circ$

ANALYSIS:

Consider the free-body diagram of the problem in Figure 1.7(b). The moment vector applied by the screwdriver aligns with the screwdriver. To compare this with the moment required about the axis of the screw, the appropriate component of that moment is necessary. Following the discussion in Section 1.1.1, the dot product can be used to determine this projection.

The dot product defines the *scaled* projection of one vector in the direction of another. Using (1.8), the component of the screwdriver moment oriented along the axis of the screw is given by:

$$M^* = \mathbf{M}_{driver} \cdot \mathbf{u}_{screw} = \|\mathbf{M}_{driver}\| \cos(30^\circ) = 0.866\|\mathbf{M}_{driver}\| \quad (1.21)$$

From static equilibrium, this must balance with the reaction moment of the screw (negative because we are trying to unscrew it).

$$\begin{aligned} \sum M = 0 &= -M^* + M_{screw} = -(0.866\|\mathbf{M}_{driver}\|) + (10) \\ \implies \|\mathbf{M}_{driver}\| &= 11.55 \text{ N}\cdot\text{m} \quad (1.22) \end{aligned}$$

Therefore you must apply a moment of 11.55 N-m to the screwdriver to break the screw loose.

Example 1.3 (2D Static Equilibrium).

PROBLEM STATEMENT: From the free-body diagram shown in Figure 1.8, solve for the unknown forces F_2 and F_3 .

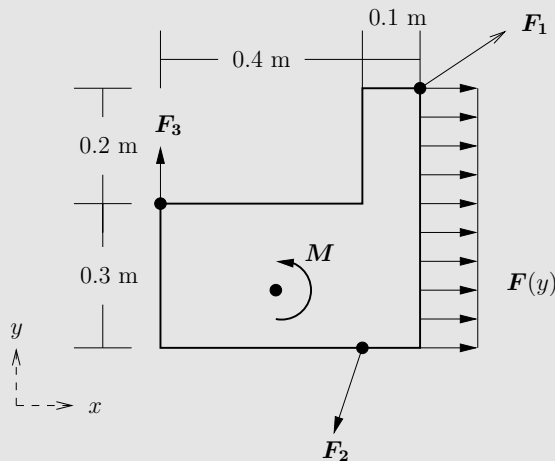


Figure 1.8: Example 1.3

KNOWNs:

For this problem, the following information is given:

- The force F_1 has a magnitude of 225 N oriented at 30° counter-clockwise from the x -axis.
- The moment M has a magnitude of 150 N-m oriented parallel to the z -axis.
- The distributed load $F(y)$ is uniform across the entire right face of the object with a magnitude of 200 N/m.
- The force F_3 is oriented vertically.

ANALYSIS:

To solve this problem, we will need to apply the 2D conditions of static equilibrium:

$$\begin{aligned}\sum F_x &= 0 \\ \sum F_y &= 0 \\ \sum M_{Oz} &= 0\end{aligned}\tag{1.23}$$

Each of the loads on the body must be expressed in the appropriate form. Begin with reducing the distributed load to a statically equivalent point load.

$$\mathbf{F}_{eq} = \int_0^{0.5 \text{ m}} \mathbf{F}(y) dy = (200 \text{ N/m})(0.5 \text{ m}) = 100 \text{ N}\tag{1.24}$$

Because the load is uniformly distributed, this point force acts at the center of the distributed force, $(x, y) = (0.5, 0.25) \text{ m}$.

Now decompose all forces and moments into their respective x -, y -, and z -components.

$$\mathbf{F}_1 = \begin{Bmatrix} F_{1x} \\ F_{1y} \end{Bmatrix} = \begin{Bmatrix} 225 \cos(30^\circ) \\ 225 \sin(30^\circ) \end{Bmatrix} = \begin{Bmatrix} 195 \\ 113 \end{Bmatrix} \text{ N}\tag{1.25}$$

$$\mathbf{F}_{eq} = \begin{Bmatrix} F_{eqx} \\ F_{eqy} \end{Bmatrix} = \begin{Bmatrix} 100 \\ 0 \end{Bmatrix} \text{ N}\tag{1.26}$$

$$\mathbf{F}_2 = \begin{Bmatrix} F_{2x} \\ F_{2y} \end{Bmatrix}\tag{1.27}$$

$$\mathbf{F}_3 = \begin{Bmatrix} F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_{3y} \end{Bmatrix}\tag{1.28}$$

$$\mathbf{M} = \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 150 \end{Bmatrix} \text{ N-m}\tag{1.29}$$

Note that while the moment appears 3D, it is in fact necessary for the moment vector to point entirely out-of-plane for this problem to reduce to the 2D case.

With each term appropriately defined, use the equations of static equilibrium to solve. There is only one unknown x -term, so begin with that condition.

$$\begin{aligned}\sum F_x = 0 &= F_{1x} + F_{eqx} + F_{2x} + F_{3x} = (195) + (100) + F_{2x} + (0) \\ &\implies F_{2x} = -295 \text{ N}\end{aligned}\tag{1.30}$$

There are two unknown y -forces. The y -force equilibrium alone will be insufficient to solve for either one. Instead begin with moment equilibrium. By

considering the sum of moments about The point of application of one of the forces, that unknown term is eliminated from the equation (distance from a point to itself is zero, so it does not contribute to the moment). Using equation (1.15) to define moments, take the sum of moments in the z -direction about the point where \mathbf{F}_3 is applied.

$$\begin{aligned}\sum M_{O3,z} = 0 &= F_{1y}d_{1x} + F_{2y}d_{2x} - F_{1x}d_{1y} - F_{2x}d_{2y} - F_{eqx}d_{eqy} + M_z \\ &= (113)(0.5) + (F_{2y})(0.4) - (195)(0.2) - (-295)(-0.3) \\ &\quad - (100)(-0.05) + 150 \\ &= 0.4F_{2y} + 84 \\ \implies F_{2y} &= -210 \text{ N}\end{aligned}\tag{1.31}$$

With only one unknown y -force remaining, direct application of the y -equilibrium equation completes the analysis.

$$\begin{aligned}\sum F_y = 0 &= F_{1y} + F_{eqy} + F_{2y} + F_{3y} = (113) + (0) + (-210) + F_{3y} \\ \implies F_{3y} &= 97 \text{ N}\end{aligned}\tag{1.32}$$

The unknown forces are now completely defined.

DISCUSSION:

- The location of the moment was not given in the problem because it is not important. An applied moment will have the same effect on static equilibrium regardless of where it acts.
- The x - and y -components of each force are positive in the positive x - and y -directions (right and up). Even though the \mathbf{F}_2 vector is pointed down and left in the free-body diagram, its components are both negative. The negative sign here does not indicate that \mathbf{F}_2 points in the opposite direction as shown, because the signs are specifically attached to components that are defined in the traditional manner.
- When using equation (1.15) to define moments, take care with signs. Distances are measure from the point where the moment is computed to the force. A force to the right of the point would have a positive distance, for example.

1.1.3 Boundary Conditions and Static Determinacy

Physical situations impose forces and moments in many ways. In general, these are classified as **actions**, or applied loads, and **reactions**, the body's

response to the actions. For example, the weight of a student is typically considered an action, while the force of the floor pushing upward in response is the reaction. Typical static analysis involves determination of the reactions. In engineering practice, **supports** are used to provide specific reactions to the body of interest.

These supports provide **boundary conditions** to a statics problem. That is, there is some information known about the support before solving the problem. Physically, supports come in a nearly infinite variety, but we are able to represent these as a few simple models. Choosing the correct model is of critical importance, however, as this will dictate the kind of reactions provided to the problem. Table 1.1 lists some common supports.

Table 1.1: Some common model supports and associated reactions

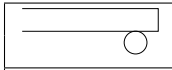
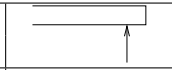
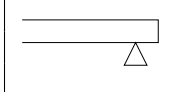
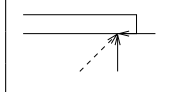
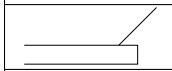
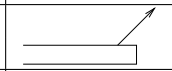

		<ul style="list-style-type: none"> • 2D roller support • Provides force normal to roller
		<ul style="list-style-type: none"> • 2D pin support • Provides a force with unknown magnitude and direction • Typically decomposed into unknown x- and y-forces
		<ul style="list-style-type: none"> • Cable support (2D and 3D) • Provides <i>tensile</i> force along cable
		<ul style="list-style-type: none"> • Fixed support (2D and 3D) • Provides all possible reactions

Table 1.1 is not exhaustive, but the reactions provided by any support can be easily established by determining what kind of motion the support restricts. For example, a roller support prevents displacement in the direction normal to the roller, but places no restrictions on movement in the tangential direction or rotation about the roller; it therefore provides a reaction force normal to the roller, but no tangential force or moments about the roller. Similarly, a pin resists displacement in any direction but allows free rotation, so it provides a force vector in any direction (or unknown x - and y -force components) but no moment.

While the nature of the reactions is known based on the associated motion restrictions, their precise values are not prior to solution of a statics problem. This can present additional issues in many cases. Consider a 2D cantilevered beam (Figure 1.9(a)).

The cantilever provides an x -force, a y -force, and a z -moment. In two dimensions, we have three equilibrium equations that can be used to solve for these unknown reactions. If an additional roller is included at the right end (Figure 1.9(b)), then there is one more unknown force. Unfortunately, static equilibrium does not provide any additional information, so this problem is currently unsolvable. A problem in which there are more unknowns than usable equations of static equilibrium is called **statically indeterminate**. For k applicable equations and m unknowns,

$$n = m - k \quad (1.33)$$

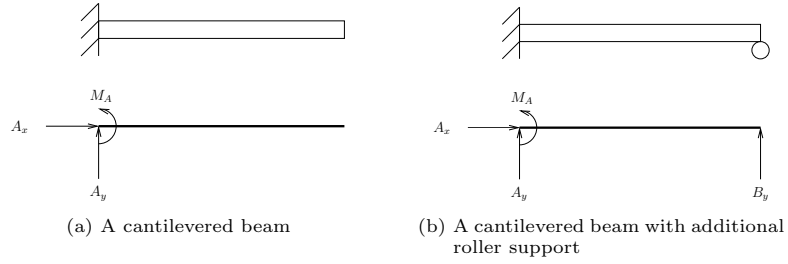


Figure 1.9: Effect of supports on static determinacy

is the **degree of static indeterminacy**. When $n = 0$, the problem may be solved using static equilibrium and is termed **statically determinate**. Some emphasis must be placed on *usable* equations. A 2D beam supported by three rollers (shown in Figure 1.10), has three unknowns and there are three equations of equilibrium in 2D. However, the force balance in the x -direction is not applicable as there are no x -forces. In this case, it appears that $m = 3$ and $k = 2$, so the beam is statically indeterminate in the first degree. However, one could use the y -force equilibrium equation as well as moment balances about two of the supports to solve for the reactions; take care to remember that although the 2D equilibrium model typically consists of two force equations and a moment equation, it is not necessarily required to follow that pattern.

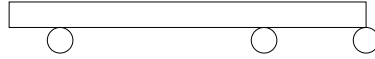


Figure 1.10: A beam supported by three rollers

1.1.4 Internal Forces

In the study of solid mechanics, we consider only **continua**, materials that exist as a single body. Force information must be transmitted across the object between the points of action and reaction. Consider the bar shown in Figure 1.11(a). The applied force at the right end is balanced by a force of equal magnitude at the support. A real material is not a true continuum, however. It consists of a multitude of discrete particles, as illustrated by Figure 1.11(b). If the solid is static, then every particle that makes it up must also be static; the equations of static equilibrium are satisfied at each point in the material. This is the means through which load information is passed through the material.

With the continuum assumption, we never consider discrete particles. The concept of inter-particle forces is still useful however. If we were to “look inside” the material, whatever surface we consider must be in equilibrium with the rest of the material. In order for this to be true, there must be **internal forces** on the surface. These internal forces are of critical importance in the analysis of deformable solids because they are local to a point in the body, and can therefore

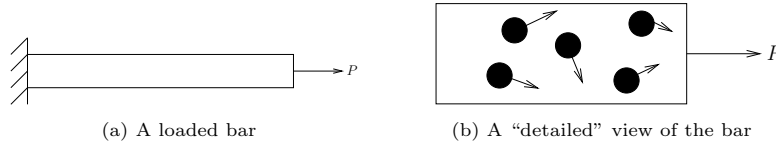


Figure 1.11: A continuum model of a bar

be related to the deformation at that point. To determine internal forces, we take **sections** or slices of the material, and consider equilibrium of that section with the external loads. Physically, internal loads are distributed forces, but it is often convenient to model them as single point forces and moments. In many cases, especially in 2D or 3D, explicit determination of the internal loads can be quite difficult (and often unnecessary), but the *concept* of internal forces is central to the field of solid mechanics. Specific techniques for determining internal loads of various problem classes are presented as appropriate throughout the text.

1.1.5 Geometry

Several geometric and material properties are very commonly used in mechanical analysis. A few of the most important are discussed in the following.

1.1.5.1 Centroids, Centers of Mass, and Centers of Gravity

Consider a generic finite body, as shown in Figure 1.12. The **centroid** of this body, denoted $(\bar{x}, \bar{y}, \bar{z})$, is its geometric center and is found by determining the mean location of every point in the body. Each point in this body can be represented as a small volume ΔV with a location. If ΔV is symmetric, then it is trivial to define the center of that volume $(\tilde{x}, \tilde{y}, \tilde{z})$. If we consider infinitesimal volumes, then $\Delta V \rightarrow dV$ and the volume is a single point, so $(\tilde{x}, \tilde{y}, \tilde{z}) \rightarrow (x, y, z)$. Taking the arithmetic mean of these locations:

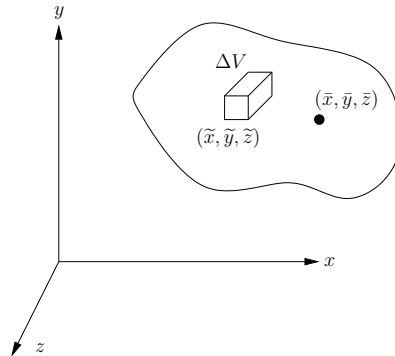


Figure 1.12: The point distribution in a finite volume

$$\bar{\mathbf{x}} = \frac{\int \mathbf{x} dV}{V} \quad (1.34)$$

$$\bar{x} = \frac{\int x dV}{V} \quad \bar{y} = \frac{\int y dV}{V} \quad \bar{z} = \frac{\int z dV}{V} \quad (1.35)$$

Or in 2D:

$$\bar{x} = \frac{\int x dA}{A} \quad \bar{y} = \frac{\int y dA}{A} \quad (1.36)$$

where A and dA are areas rather than volumes.

A related, but not entirely equivalent, property is the **center of mass** $\bar{\mathbf{x}}_m$. The center of mass is the mean location of the mass of a body, and therefore is not purely a geometric property. Following a similar procedure to that used for centroids, the arithmetic mean of the masses of every point is given by:

$$\bar{\mathbf{x}}_m = \frac{\int \mathbf{x} dm}{m} \quad (1.37)$$

A third related property is the **center of gravity**, denoted $\bar{\mathbf{x}}_g$ or \mathbf{c}_g . Similar to the previous two measures, the center of gravity is an average location, but in this case it is the mean location of the weights of the particles. Weight is dependent on volume (geometry), density (material), and the gravitational field (environment), making a much more complex property. Following exactly the same process:

$$\bar{\mathbf{x}}_g = \frac{\int \mathbf{x} dW}{W} \quad (1.38)$$

For most practical applications, the gravitational field is uniform around a body. In such a case, $dW = g dm$, where the gravitational acceleration g is a constant. Substituting this into (1.38):

$$\bar{\mathbf{x}}_g = \frac{\int \mathbf{x} g dm}{gm} = \frac{g \int \mathbf{x} dm}{gm} = \frac{\int \mathbf{x} dm}{m} = \bar{\mathbf{x}}_m \quad (1.39)$$

In a uniform gravitational field, the center of gravity reduces to the center of mass. The mass of a differential volume dm is equal to the density of the particle multiplied by its volume, ρdV . If the material is **homogeneous** then its properties, including density, do not vary in the body. Substituting the differential mass into (1.37):

$$\bar{\mathbf{x}}_m = \frac{\int \mathbf{x} \rho dV}{\rho V} = \frac{\rho \int \mathbf{x} dV}{\rho V} = \frac{\int \mathbf{x} dV}{V} = \bar{\mathbf{x}} \quad (1.40)$$

For homogeneous materials, the center of mass reduces to the centroid. Under appropriate conditions, all three of these properties are equivalent. The necessary conditions (homogeneity of the material and uniformity of the gravitational field) are very realistic for many practical applications, so for our purposes it is sufficient to consider only the centroid currently.

The integral definition of the centroid is precise and applicable to any finite body. However, it is not necessarily trivial to compute. In some cases, it may

be possible (and desirable) to compute the centroid without integrating. Recall that a finite subvolume ΔV of the body has its own centroid, (\tilde{x}, \tilde{y}) . If this is precisely known, then the mean location of points in ΔV is already defined. If a finite body can be decomposed into a finite number of ΔV , then computing the mean location of those ΔV is perfectly equivalent to using an integral to take the mean of every point in the body. Consider a finite number of ΔV and take the arithmetic mean of their locations:

$$\bar{x} = \frac{\sum \tilde{x}_i \Delta V_i}{\sum \Delta V_i} \quad (1.41)$$

A **composite body** is one made of simpler shapes. We can isolate each shape, easily determine its local centroid and volume (or area in 2D), and use equation (1.41) to determine the composite centroid without ever considering the integral. A table of the centroids of some simple geometries is included in Appendix A.

1.1.5.2 First and Second Area Moments

Two additional geometric properties of areas that are useful in mechanical analysis are the **first and second moments of area**. These two properties measure the distribution of matter over an area. Their precise use will be discussed later in the text as appropriate; the current discussion is focused on quantifying them. Consider the generic area shown in Figure 1.13. Definition of the coordinate system is necessary for clear definitions of the first and second area moments. The coordinate system is chosen such that the *origin is at the centroid*.

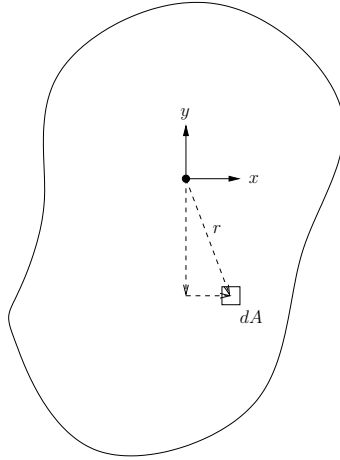


Figure 1.13: Determination of first and second area moments

Consider the first area moment:

$$Q = \int x dA \quad (1.42)$$

This is recognizable as the numerator from the definition of the centroid. x measures distance from the origin. For practical applications, we are typically interested in the distribution of matter about the x -axis, so consider only the distance from that axis, measured by y .

$$Q_x = \int y dA \quad (1.43)$$

Matter is distributed both above and below the x -axis, represented by the sign of y . If this integral is considered over the entire area of the body, then $Q_x = 0$ because our origin is the centroid: matter is evenly distributed about this point by definition. However, if we choose to measure Q_x over only part of the area, we get a non-zero value. Following the discussion in the previous section, this is a non-trivial calculation, but by considering composite bodies, it is possible to reduce (1.43) to a simple algebraic expression:

$$Q_x = \tilde{A} \cdot \tilde{d}_y \quad (1.44)$$

where \tilde{A} is the area for which the first area moment is being computed (not necessarily the total area of the object) and \tilde{d}_y is the distance from the centroid of \tilde{A} to the x -axis, illustrated in Figure 1.13. Using this formula, it is simple to see that Q_x for the entire area must be zero, as \tilde{d}_y is zero in that case.

Similarly, the second area moment is defined with respect to each axis. The second area moment about the x -axis is:

$$I_{xx} = \int y^2 dA \quad (1.45)$$

This quantity also measures the distribution of matter, but note that y is squared, and therefore always positive. There is no sign associated with the second area moment. It is a measure of the distance of matter from the axis, without consideration for direction. The expression given in equation (1.45) may look familiar; if multiplied by density, this would be the mass moment of inertia that appears in rotational motion. For this reason, the second area moment is often called the **area moment of inertia**. Keep in mind that this is strictly a geometric property, however, with no consideration given to the material.

Once again, integrating this expression can be challenging. Our options for avoiding that work are more limited in this case, but fortunately others have already performed the integrals for many common geometries (some are listed in Appendix A). For example, for the two simple geometries shown in Figure 1.14, the second area moments about the x -axes are given by:

$$\begin{aligned} \text{Rectangle: } I_{xx} &= \frac{bh^3}{12} \\ \text{Circle: } I_{xx} &= \frac{\pi D^4}{64} \end{aligned} \quad (1.46)$$

While second area moment information is available for many simple shapes, there are infinite possible configurations, so tables such as those in Appendix

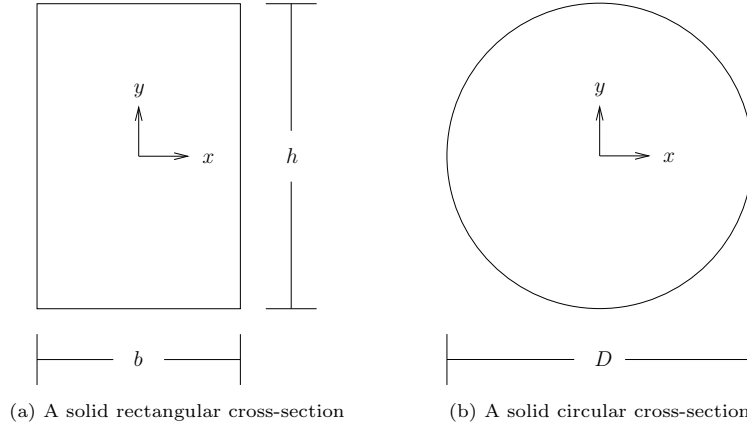


Figure 1.14: Common areas, with coordinate systems placed at the centroids

A will never be sufficient. There is one very powerful tool that allows us to continue taking advantage of tables even for complex bodies. The **parallel axis theorem** states that the second area moment of a body about its own axis can be related to the second area moment about any other axis parallel to the first.

$$I_1 = I + Ad^2 \quad (1.47)$$

I is the second area moment computed about one axis (typically one of the coordinate axes), A is the area of the shape, and d is the distance from this axis to the 1-axis where we want to know the second area moment. If we consider a composite shape, then:

$$I_{xx} = \sum (I_i + A_i d_i^2) \quad (1.48)$$

The second area moment about the global x -axis is found by determining the second area moment of each subdomain about that axis using parallel axis theorem, and summing them.

A related quantity is the **polar second area moment**. Rather than perpendicular distance to an axis, the polar second area moment measure total distance to a single point:

$$J = \int r^2 dA \quad (1.49)$$

r is the distance from the origin of the coordinate system. As shown in Figure 1.13, $r^2 = x^2 + y^2$ by Pythagorean theorem. Substituting this definition into

(1.49):

$$\begin{aligned}
 J &= \int r^2 dA \\
 &= \int (x^2 + y^2) dA \\
 &= \int x^2 dA + \int y^2 dA \\
 &= I_{yy} + I_{xx}
 \end{aligned}
 \tag{1.50}$$

In this context, I_{xx} and I_{yy} are sometimes called the **planar second area moments**. The polar second area moment of a plane is the sum of the planar second area moments.

Example 1.4 (Geometric Properties).

PROBLEM STATEMENT:

Consider the T-bracket shown in Figure 1.15. Determine the centroid of the shape and second area moment of the shape about the x -axis. The origin of the coordinate system is the centroid of the shape.

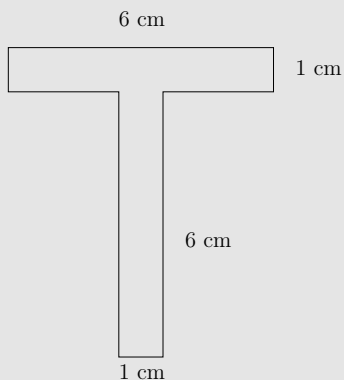


Figure 1.15: Example 1.4

KNOWNs:

- Dimensions are as shown in Figure 1.15

ANALYSIS:

Both centroid and second area moment are purely geometric properties, so only the shape is required for their determination (no material properties, etc.).

First, note that the bracket is symmetric left to right. The x -component of the centroid must be precisely in the center. To determine the y -component of the centroid, it is perhaps simplest to decompose the T into two rectangles.

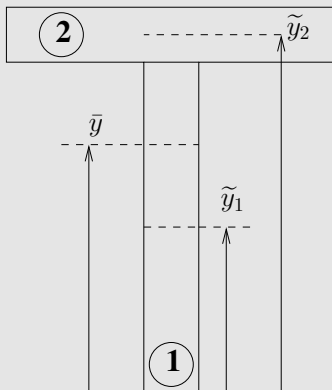


Figure 1.16: Shape decomposition

From (1.41) (in terms of area rather than volume), we will need the centroid and area of each subdomain. The centroids of the decomposed shapes must be measured in the same \tilde{y} -coordinate system, referencing the same datum. Since the rectangles are symmetric in the y -direction, we observe that their respective centroids are:

$$\begin{aligned}\tilde{y}_1 &= 3 \text{ cm} \\ \tilde{y}_2 &= 6 + 0.5 = 6.5 \text{ cm}\end{aligned}\tag{1.51}$$

measured from the bottom of the T. To make use of equation (1.41), the local centroids must be weighted by the areas of their subdomains. In this case, they are the same:

$$A_1 = A_2 = 6 \text{ cm}^2\tag{1.52}$$

$$\bar{y} = \frac{A_1 \tilde{y}_1 + A_2 \tilde{y}_2}{A_1 + A_2} = 4.75 \text{ cm}\tag{1.53}$$

A similar technique and much of the same information is used to determine the second area moment. The x -axis passes through the centroid defined by (1.53). Since the shape is not a simple shape, the same decomposition will be used to determine the necessary properties. The only new piece of information required is the local second area moment of each subdomain. From equation (1.46), I for each subdomain is:

$$\begin{aligned}I_1 &= \frac{(1)(6)^3}{12} = 18 \text{ cm}^4 \\ I_2 &= \frac{(6)(1)^3}{12} = 0.5 \text{ cm}^4\end{aligned}\tag{1.54}$$

Equation (1.48) also requires a distance d . This is the distance between local and global centroids, $\bar{y} - \tilde{y}$. Substituting all relevant data into (1.48):

$$I_{xx} = (18 + (6)(3 - 4.75)^2) + (0.5 + (6)(6.5 - 4.75)^2) = 59.44 \text{ cm}^4 \quad (1.55)$$

1.2 Deformable Body Mechanics

Static analysis is concerned primarily with the forces acting on a body, with no regard to any deformation of the body. In fact, any change of shape or dimension of a body subjected to loads has been completely neglected. Recall that a continuum is an assembly of particles; consider a simple two-particle system as shown in Figure 1.17(a). If this system is a **rigid body**, it may displace (Figure 1.17(b)) or rotate (Figure 1.17(c)), but the distance between the particles will not change. If the body **deforms**, then the particles displace relative to each other (Figure 1.17(d)). In a continuum we do not consider discrete particles, but the relative motion still defines deformation. The equations of static equilibrium still hold for a deformable body, but are insufficient to completely describe it.

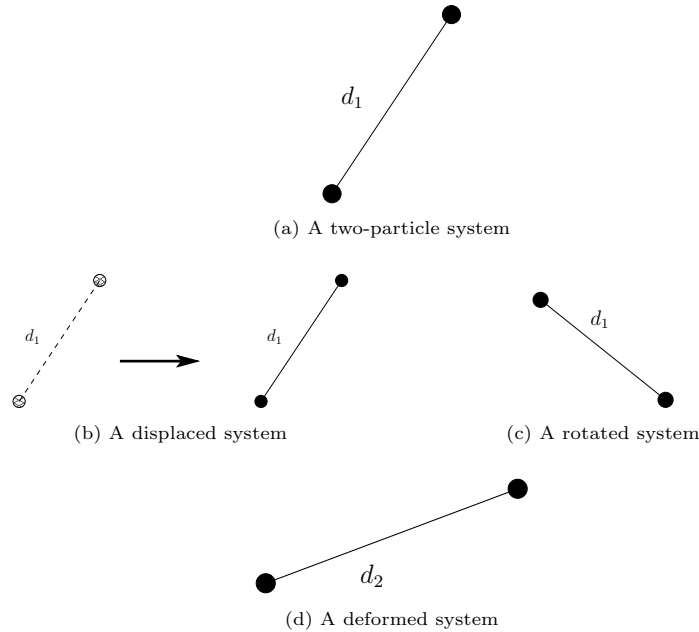


Figure 1.17: Comparison of rigid and deformable body motion

1.2.1 Stress and Strain

For a thorough analysis of deformable body mechanics to take place, we must have a well-defined and suitable measure of deformation. Consider the simple body shown in Figure 1.18(a). At this point we know that deformation refers to the internal motion of a body, so displacement of this simple bar (Figure 1.18(b)) is not adequate to describe the deformation. If we were somehow able to measure the displacements of every single particle and compare them to each other (Figure 1.18(c)), then deformation would be fully-defined, albeit not usefully when we could be dealing with billions of discrete points. An aggregate measure of the particle displacements that holds in a continuum sense could be exactly what is needed. For our simple body, the *change of length* is that aggregate measure (Figure 1.18(d)); the entire body may displace, but our only concern is the displacement of one end with respect to the other.

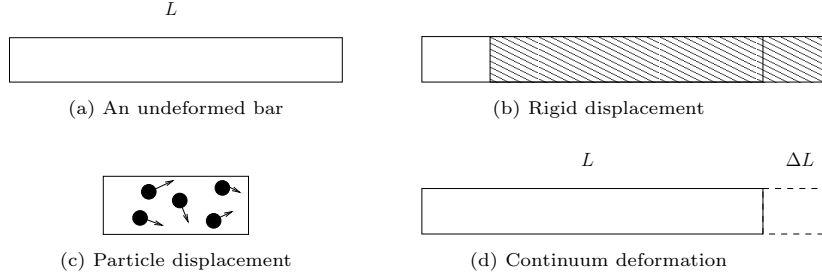


Figure 1.18: Measures of deformation

A simple example shows, however, that even the change of length is insufficient as a general measure of deformation. Consider two bars, one with initial length $L_1 = 1$ m and the second with initial length $L_2 = 100$ m. Each bar is subjected to a change of length of $\Delta L = 0.1$ m. The same change of length has profoundly different consequences in the two bars. In the first, it is a 10% change, while the second bar experiences only a 0.1% change. Obviously ΔL cannot represent the entirety of the physics, but this experiment leads us directly to an improved measure. The percent change of length considers the initial configuration with the change of length. We formalize this concept through the idea of **normal strain**. Given a material fiber (a single material point) of length ds that deforms to a length ds' , the normal strain is defined:

$$\varepsilon = \frac{ds' - ds}{ds} \quad (1.56)$$

The strain is a dimensionless, relative measure of deformation. A positive normal strain is associated with increased length, and negative strain with decreased length.

The normal strain alone is still not enough to fully define the deformation of a body. The normal strain measures change of length (or area, or volume, if used correctly), but it does not capture a change of shape. For that, an

additional definition is required. Given two perpendicular material fibers, the **shear strain** is the *change in angle* between the fibers following deformation (not the new angle between the fibers). The sign convention for shear strains is illustrated in Figure 1.19. For two lines oriented as shown in Figure 1.19, the shear strain γ between them is positive if the angle reduces, and negative if the angle increases. Shear strain is also dimensionless, so is always measured in radians.

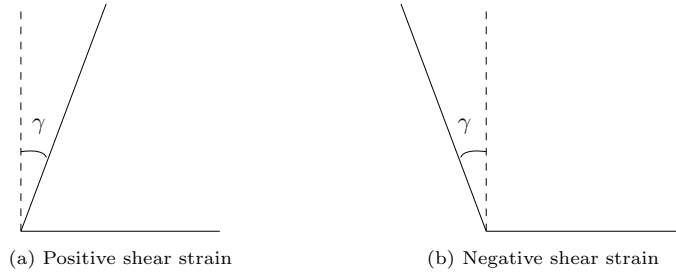


Figure 1.19: Shear strain sign conventions

Strains can be considered an internal measure of displacement, the relative motion of material particles. Logically, those internal displacements would be related to internal forces. It follows that the measure of these internal forces is as critical to the study of mechanics as strains. Let us begin with another thought experiment. Consider two bars again. The first bar has a 1 mm diameter, and the second has a 1 m diameter. Both bars are subjected to 10 N load. This same force has a much more “noticeable” effect on the smaller bar than the larger. A measure of force that scales with the size of the body is necessary.

A finite body subjected to external loads is shown in Figure 1.20. Any part of this body carries internal forces. These forces are distributed over the surface of the piece of material and may point in any direction. If a small enough area ΔA is considered, the distributed force is effectively a single point load ΔF . In the limiting case, this can be used to measure the internal force at a single point. We use this to define the **stress**:

$$\sigma = \lim_{\Delta A \rightarrow 0} \frac{\Delta F}{\Delta A} \quad (1.57)$$

The dimensions of stress are force per unit area.

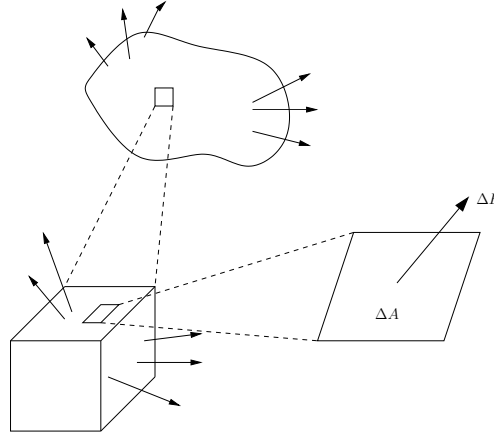


Figure 1.20: External and internal forces

The force vector may be oriented in any direction, but can be decomposed into components normal and tangential to the area. The stress defined using the normal force is the **normal stress**, and the stresses defined using the tangential forces are the **shear stresses**. Stresses and strains are not vectors, they are **tensors**, quantities that transform in certain ways when the coordinate system is changed. A cubic material point aligned with the coordinate axes is shown in Figure 1.21. The internal forces on each face of this cube, measured as stresses, are decomposed into normal and shear components. Note the subscripts: the first subscript describes where a stress is acting, and the second details its direction. The sign of the stress is the combination of the two. For example:

- $\sigma_{xx} \implies$ The normal stress on the x -face in the x -direction
- $\tau_{xy} \implies$ The shear stress on the x -face in the y -direction
- $\tau_{zx} \implies$ The shear stress on the z -face in the x -direction

The normal stresses are denoted by σ and the shear stresses by τ . The stresses on the opposite faces of the cube are equivalent; the normal stress on the negative x -face in the negative x -direction is positive σ_{xx} .

One additional comment on this notation: because shear stresses and normal stresses use different symbols (τ vs. σ), we always know the nature of the stress even without the subscripts. For this reason, it is common to use only a single subscript on a normal stress; if a normal stress is on the x -face, then it must point in the x -direction (and vice versa). Thus, σ_{xx} and σ_x are equivalent. The double subscript is used throughout this text, but it is not uncommon to see the single subscript notation in other references.

The three visible faces of the cube in Figure 1.21 show nine independent stress components that can be expressed as a matrix:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \quad (1.58)$$

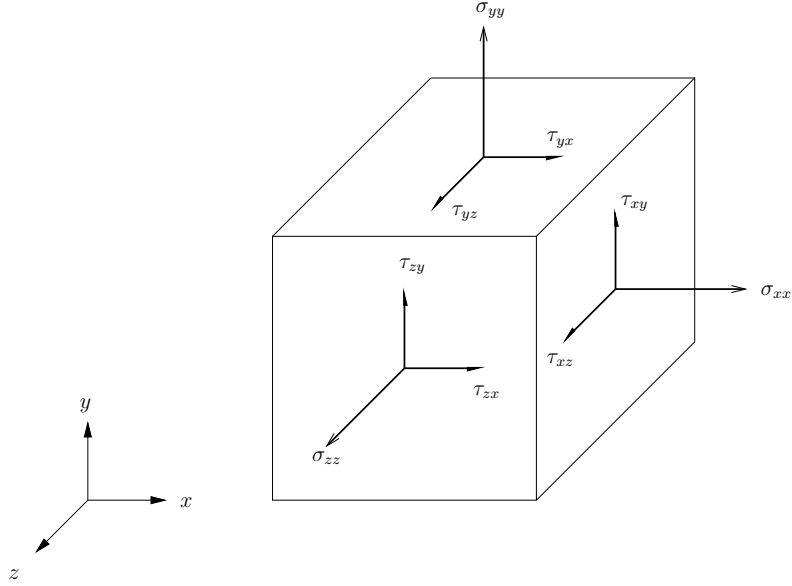


Figure 1.21: A complete state of stress

Fortunately, this simplifies somewhat. Through careful application of the balance of angular momenta, it can be shown that opposing shear stresses are equal; that is, the stress tensor is symmetric, resulting in only six independent stresses:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \quad (1.59)$$

Similarly, there are nine strains: normal strains ε and shear strains γ . These strains can also be shown to be symmetric, reducing to a tensor of six independent components:

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_{zz} \end{bmatrix} \quad (1.60)$$

In practical application, the stresses and strains are often presented as **quasivectors**. The components are organized as vectors, but are actually tensors. This representation is known as **Voigt's notation**, and is a very useful tool in 2D and 3D analysis.

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{array} \right\}; \quad \left\{ \begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{array} \right\} \quad (1.61)$$

Looking back on our experiments, we can clearly observe that a single value of stress or strain is much more meaningful than total change of length or force. A strain value of $\varepsilon = 0.01$ relates to different total changes of length for the two bars, but it represents the same physical effect on each bar; a 1% change of its length. Likewise, a given value of the stress σ measures the same internal force per particle in each of the two bars. The larger cross-sectional area may have more particles, so a greater total force, but each point of the material is exposed to the same load in the two cases.

1.2.2 Average Stress and Strain

Detailed stresses and strains are often difficult to determine. It may suffice in some cases to consider average values instead. In taking average values of these quantities, we avoid pointwise definitions by considering only total areas and lengths. Denote the internal normal force acting on an area A as N and the internal shear force as V , then the average stresses on that area are:

$$\sigma_{avg} = \frac{N}{A} \quad (1.62)$$

$$\tau_{avg} = \frac{V}{A} \quad (1.63)$$

At first glance this may seem to defeat the purpose of defining our stresses using the limit, but average stresses can still provide very useful information if used appropriately. Likewise, the average normal strain is defined using the total length of a segment of material L and the total change of that length ΔL :

$$\varepsilon_{avg} = \frac{\Delta L}{L} \quad (1.64)$$

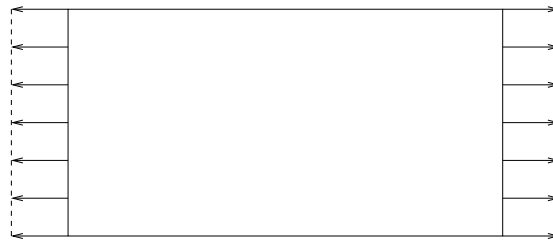
If the internal forces are uniform across an area, then every point in that area supports the same load. In such a case, the average stress is equivalent to the stress. Likewise, if a body deforms uniformly, then each point in the body experiences the same local displacement relative to its neighbors, and the average normal strain is equivalent to the normal strain. It should be noted that these average measures are in no way adequate when the body deforms substantially. Additional discussion to this point is given in a later section.

1.2.2.1 Stress Concentrations and Saint-Venant's Principle

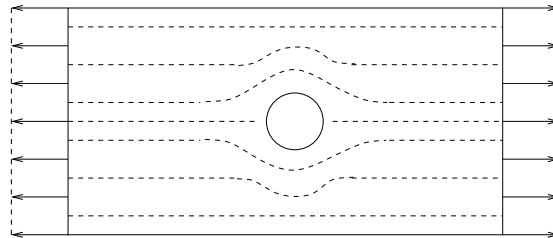
While average stress can be a useful measure in many cases, there are also many situations in which it can produce significant inaccuracies. Consider a simple plate as shown in Figure 1.22(a). The plate is uniformly loaded, so the average stress $\frac{F}{A}$ is an accurate measure of the stress in the entire body. However, if the geometry changes in the body, as in Figure 1.22(b), the force distribution becomes non-uniform internally. Orientations and magnitudes of internal forces may change drastically. Even at a point where the internal forces are oriented in the same direction as the applied load, the actual stresses may

be much higher than the average stress at that location. In the case of a large plate with a circular hole, the stress at the top and bottom edges of the hole is three times higher than the average stress (Figure 1.22(c)). This phenomenon is known as **stress concentration**. Any change in geometry will cause a stress concentration; sharper changes exaggerate the phenomenon further. **Stress concentration factors** k have been determined for many different situations and may be found in various references and design guides. These factors allow for straightforward estimation of the actual stresses at stress concentrations:

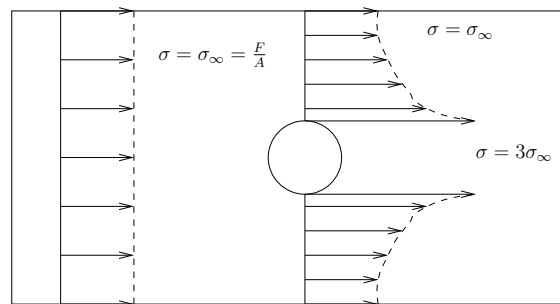
$$\sigma = k\sigma_{avg} \quad (1.65)$$



(a) A uniformly loaded plate



(b) Internal force in a plate with a hole



(c) Stress distribution across a plate with a hole

Figure 1.22: Development of Stress Concentrations

The use of stress concentrations is primarily a design factor and detailed discussion is beyond the scope of this text. In many cases it is possible to

avoid them entirely. Consider a simple bar that is loaded non-uniformly at the end. This load may be due to realistic supports or intentional design. As distance from the non-uniformity is increased, the magnitude of the non-uniformity internal to the body decreases. Greater than some limiting distance from the non-uniformity, the internal loads have equilibrated and become uniformly distributed across the cross-section of the material (Figure 1.22(c)). The non-uniformities due to supports are generally called **end effects**, but they are not the only source; any change in geometry or loading creates at least local disturbances. At any distance further from the non-uniformity than the critical distance, mechanical analysis can accurately reduce to a simpler uniform case. This is known as **Saint-Venant's principle**. Keep in mind that the distance is not well-defined; it will vary with problem specifics. However, the idea that simpler analyses are possible at a distance from non-uniformities is very useful.

1.2.3 Mechanical Properties of Materials

As force is related to displacement, stress is related to strain. The precise nature of this relation is not universal. The resistance of a particle to motion may depend on the strength of its bonds with its neighbors or the specific types of bonds, which in turn may depend on the temperature, the rate of loading, the magnitude of deformation, etc. Regardless of complexity, the inter-particle bonding behavior is a material property; a material is largely defined by this behavior. Because the relations between stress and strain depend on the constitution of the matter, they are termed **constitutive laws** or **constitutive theories**. As stated previously, these relations may be very complicated. To simplify matters currently, the remainder of the work in this text will assume that a material is *static* unless specifically stated otherwise.

1.2.3.1 Elasticity and Plasticity

All real materials deform when subjected to loads. This deformation, regardless of scale, can take several forms. It may be a change of size, or a change of shape. It also may be a temporary deformation. In this case, if the load is removed, then the deformation recedes. This phenomenon is known as **elasticity** and is easily observed in many mundane situations. A student may lightly bend his pen, but when he releases it the pen springs back to its initial, straight configuration. Not all deformation is recoverable, though. If a material is deformed far enough, the deformation becomes permanent. This **plasticity** is due to microstructural damage to the material.

In some cases, it is possible to determine a material's constitutive behavior using chemistry and quantum mechanics. However, it is much more common to experimentally define the mechanical properties of a material. Consider the results of a simple experiment shown in Figure 1.23, in which a test specimen is pulled apart. Stress is plotted as a function of strain. Such a curve is generated by loading the body **quasi-statically** (slowly enough that rate effects are negligible) at room temperature to minimize variables.

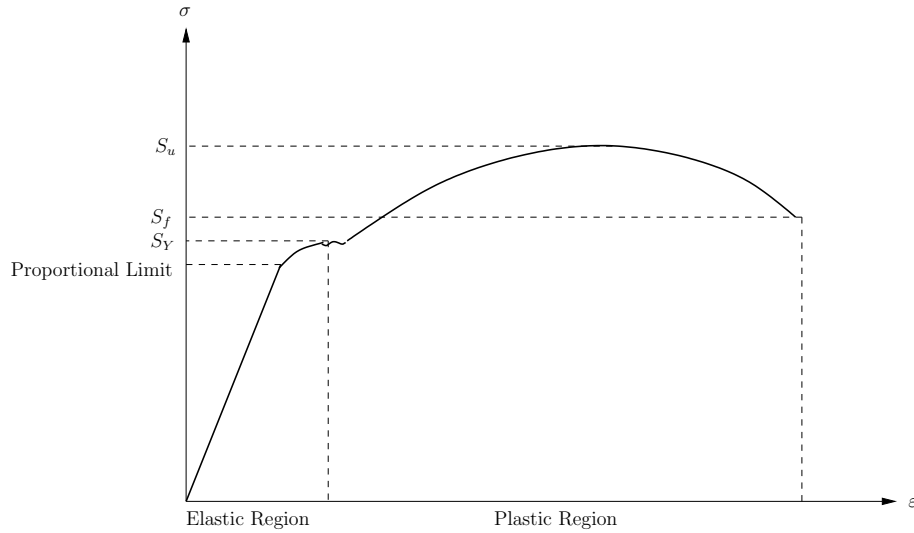


Figure 1.23: The tensile stress-strain curve

The axes of the stress-strain curve in Figure 1.23 are **engineering stress** and **engineering strain**; they are average values determined using the initial length and cross-sectional area of the test specimen, rather than an incremental or current value.

$$\sigma = \frac{P}{A_0} \quad (1.66)$$

$$\varepsilon = \frac{\Delta L}{L_0} \quad (1.67)$$

While some features are exaggerated for clarity, this curve is characteristic of many common materials, especially ductile metals. There are several key regions and points highlighted in Figure 1.23:

- **Elastic region:** The elastic region of the stress-strain curve is defined by recoverable deformation. Below some limiting stress or strain, the strain will return to zero when the material is unloaded. All materials experience at least some elasticity. In metals, the elastic region is typically quite small.
- **Plastic region:** Permanent deformation is called plastic (not to be confused with polymers). If an object is deformed beyond the elastic region, then deformation becomes nonrecoverable.
- **Elastic limit:** The elastic limit is the point at which the material transitions from elasticity to plasticity. Nominally defined in terms of stress, but difficult to determine precisely, so not shown in Figure 1.23.

- **Proportional limit:** In many materials, the stress and strain are proportional in the elastic region. The point at which the stress-strain curve becomes nonlinear for such materials is called the proportional limit. The material is still elastic for some amount of deformation beyond the proportional limit, so this point does not define the onset of plasticity.
- **Yield stress:** A point defined *by convention* as the onset of plasticity. Denoted S_y .
- **Ultimate stress:** The peak *engineering* stress the material can experience. Sometimes also called the **tensile strength** of the material. Denoted S_u .
- **Fracture stress:** The engineering stress of the material at fracture. Denoted S_f .

It may seem counterintuitive that the fracture stress of the material is lower than the ultimate stress, but this is due to how stress is measured in such an experiment. Recall that the engineering stress is the current force divided by the original, undeformed area of the test specimen. This curve shows that when the ultimate stress is reached, enough damage has been done to the material that less force is required to continue elongation. Further plastic deformation continues to reduce the durability until fracture occurs.

In reality, when substantial elongation occurs, the cross-sectional area of the specimen reduces. The so-called **true stress** calculated using the reduced area would be higher (dividing by a smaller number). As the area continues reducing (known as **necking**, Figure 1.24), the true stress will continue to climb. The true stress at fracture is the highest value obtained. However, it is very difficult to continually measure the changing area, so engineering stress is much more common.

Focus on the elastic region (Figure 1.25). If the elastic stress-strain curve is a straight line, the material is called **linearly elastic**. The equation of a line in slope-intercept form is given by:

$$y = mx + b \tag{1.68}$$

where m is the slope and b is the y -intercept. In the case of our curve, $y = \sigma$ and $x = \varepsilon$. When the strain is zero, so is the stress, so $b = 0$. Define the slope of the stress-strain curve as E , and equation (1.68) becomes:

$$\sigma = E\varepsilon \tag{1.69}$$

Equation (1.69) is known as **Hooke's law**. Much like the equation describing linear springs with which it shares its name, Hooke's law relates the strain proportionally to the stress. The coefficient E is called the **Young's modulus**, analogous to the spring constant in Hooke's "other" law. The Young's modulus describes the amount of stress (internal force) associated with a strain

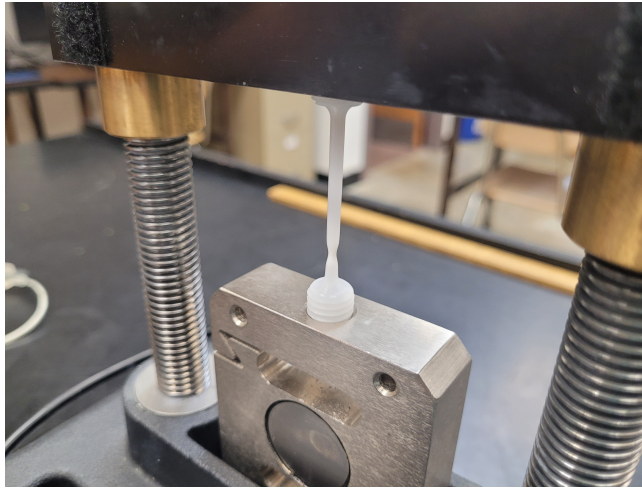


Figure 1.24: Visible necking in a tensile specimen

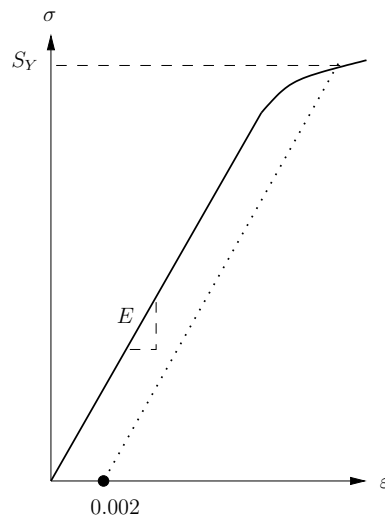


Figure 1.25: Linear elasticity

(deformation). In a sense, it is a measure of the stiffness of the material. Note that the use of stress and strain have generalized this experiment in terms of geometry; E therefore can only depend on the material itself. The Young's modulus is a **material property**. It is intrinsic to the matter and is not problem-dependent. Appendix B lists this property and others (including yield and ultimate strengths) for some common materials. Hooke's law relates stress and strain through the material, and is therefore a constitutive theory.

Take care: while Young's modulus is a deterministic material property that

will not change, Hooke's law was determined based on a 1D tensile test, and may not be fully general. Alternative experiments may reveal different constitutive relations. For example, a torsion test to relate shear stress to shear strain will show that:

$$\tau = G\gamma \quad (1.70)$$

This is another constitutive theory, also linear. G is known as the **shear modulus**, or occasionally the **modulus of rigidity**.

The Young's modulus is sometimes also referred to as the **modulus of elasticity**. To be completely clear, however, there are several moduli of elasticity that describe different modes of deformation. The Young's modulus is related to tensile deformation, the shear modulus is related to shear deformation, the **bulk modulus** to volumetric deformation, all elastic.

Pay attention to the form of the constitutive relation. Stress is a function of strain. Hooke's law is linear, so we could solve it for strain, but this is not the case in every constitutive law. This is a reflection of the physics of the problem. Stress is caused by strain. This may not be immediately obvious. Stress is a measure of force, and strain is a measure of displacement; forces cause displacements. However, we must take care with that logic. External forces displace the material, which causes strains that in turn generate stresses. Stress is the material's response to a stimulus or disturbance. For this reason, rigid bodies that cannot strain also do not incur stresses. There are certainly internal forces acting on areas, but they are not stresses.

In the simple 1D problem described by equation (1.69), a force creates a displacement (stress and strain) in the same direction. That is not the only displacement; it can be readily observed that as the material is elongated in one direction, its lateral dimensions shrink (Figure 1.26). There are no forces oriented in the lateral directions. The lateral deformation, known as the **Poisson effect**, is due to rearrangement of the microstructure following longitudinal deformation. For small deformations, it is measure by **Poisson's ratio**:

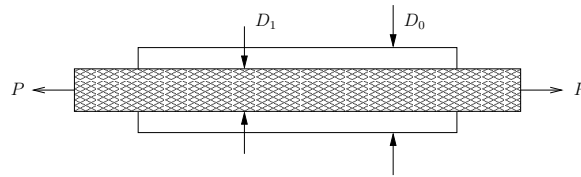


Figure 1.26: The Poisson effect

$$\nu = -\frac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}} \quad (1.71)$$

For most materials, Poisson's ratio is between 0 and 0.5. $\nu = 0$ implies that lateral and longitudinal deformation are completely decoupled. A Poisson's ratio of $\nu = 0.5$ holds for materials that do not change volume during deformation. In that case, the lateral shrinkage matches the longitudinal expansion. Due to the

negative sign in (1.71), a positive Poisson's ratio indicates materials that shrink in the lateral direction when extended in the longitudinal direction. Most materials have a positive ν , although some have been found with a negative value. Such materials, called **auxetic materials**, have a microstructure organized like origami, that unfolds and expands as it is stretched.

Much like the different moduli of elasticity, Poisson's ratio is an intrinsic material property. It is determined by the constitution of the matter independently of geometry or loading conditions. Logically, one could assume that since E , G , and ν (among others) are all dependent on the material, they may be related to each other. This is in fact the case. It can be shown that for isotropic materials (discussed in Section 1.2.3.5), only two of these properties are needed to fully define the mechanical behavior.

$$G = \frac{E}{2(1 + \nu)} \quad (1.72)$$

When the material is loaded beyond the elastic limit, its behavior changes. Most obviously, the stress and strain are no longer linearly related. As mentioned previously, the elastic limit is not easily measured. The elastic range is quite small, and the transition from linearity (called the proportional limit) is not necessarily the same as the elastic limit. The yield stress is defined to give a clear transition point for the purposes of analysis. A line parallel to the linear elastic curve is drawn at an offset of $\varepsilon = 0.002$, as shown by the light dashed line in Figure 1.25. The point where this intersects the stress-strain curve is the yield stress. Yield stress is defined by convention; it does not describe any physical property of a material. The yield stress is useful in practice though; it provides a clear point beyond which we can be sure plastic deformation is occurring. The yield stress is always greater than the proportional limit, and typically greater than the elastic limit. In most applications, a material that yields is considered to have failed. For that reason the yield stress is often called the **yield strength**, or simply the **strength**, of a material.

Plastic deformation is permanent. However, even in the plastic region not all deformation is permanent. A material loaded into the elastic region will unload along the same path. A material loaded into the plastic region will unload along a path *parallel to the elastic curve*. As shown in Figure 1.27, the material that is deformed to point **A** will return to a zero strain and zero stress state when unloaded. However, if it is deformed to point **B**, it will return to a zero stress state along a line with slope E from point **B**, ending at point **C**. The value of the strain at point **C** is the **plastic strain** or permanent deformation. The horizontal difference between points **B** and **C** is the **elastic recovery**. Note that the elastic recovery is actually greater than the elastic region of the stress-strain curve due to the increased stress difference in loaded and unloaded states; elasticity is still present even in the plastic region! If the material is loaded again, deformation will begin at point **C**, and will be linearly elastic until it reaches point **B**, at which point deformation follows the original stress-strain curve. We can effectively increase the yield strength by plastically deforming a material and unloading.

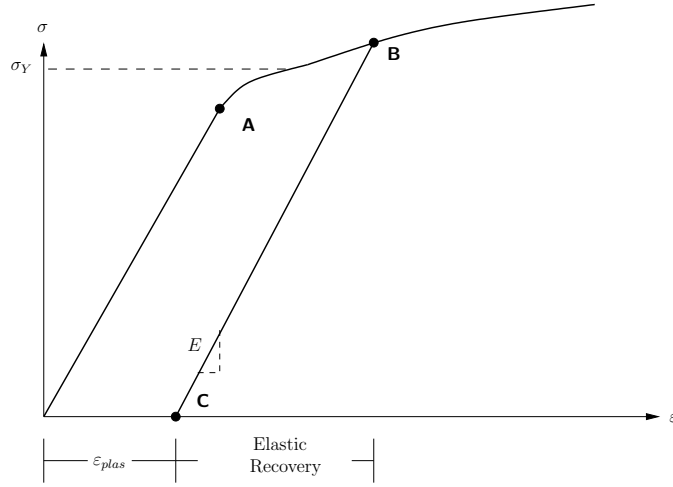


Figure 1.27: Plastic loading-unloading behavior

1.2.3.2 Generalized Hooke's Law

Hooke's law, given by equation (1.69), relates stress and strain directly.

$$\sigma = E\varepsilon \quad (1.73)$$

As mentioned previously, however, this relationship was found using a simple, one-dimensional tensile test. Assuming stress and strain are oriented in the x -direction, rewrite this relationship in terms of strain as a function of stress.

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} \quad (1.74)$$

In other words, elongation is due to the force applied in that direction. Due to Poisson effects, though, there could be other causes of deformation independent of this. If these are included, equation (1.74) becomes:

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \nu\varepsilon_{yy} - \nu\varepsilon_{zz} \quad (1.75)$$

The strains in the y - and z -direction also contribute to deformation in the x -direction. Following equation (1.74), these strains are related to the stresses in their respective directions.

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} \quad (1.76)$$

The same process may be followed to relate normal strains in the y - and z -directions to the stresses.

Shear stress and strain can be treated in the same way.

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \quad (1.77)$$

In the case of shear deformation though (for isotropic materials at least), there is nothing like a Poisson effect to create interaction between the different directions. Shear stresses are also wholly independent of normal stresses. Equation (1.72) can be substituted into the shear strain to replace G with ν and E . It is interesting to note that even though shear deformation is independent of the Poisson effect, shear stress and strain can still be related in terms of the Poisson's ratio.

$$\gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy} \quad (1.78)$$

Extending equations (1.76) and (1.78) into other directions and planes, all six strains can be related to the various stresses:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E} (\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}) \\ \varepsilon_{yy} &= \frac{1}{E} (-\nu\sigma_{xx} + \sigma_{yy} - \nu\sigma_{zz}) \\ \varepsilon_{zz} &= \frac{1}{E} (-\nu\sigma_{xx} - \nu\sigma_{yy} + \sigma_{zz}) \\ \gamma_{xy} &= \frac{2(1+\nu)}{E} \tau_{xy} \\ \gamma_{yz} &= \frac{2(1+\nu)}{E} \tau_{yz} \\ \gamma_{xz} &= \frac{2(1+\nu)}{E} \tau_{xz} \end{aligned} \quad (1.79)$$

This system of equations can be solved for stress in terms of strain.

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)} ((1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}) \\ \sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)} (\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz}) \\ \sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)} (\nu\varepsilon_{xx} + \nu\varepsilon_{yy} + (1-\nu)\varepsilon_{zz}) \\ \tau_{xy} &= \frac{E}{2(1+\nu)} \gamma_{xy} \\ \tau_{yz} &= \frac{E}{2(1+\nu)} \gamma_{yz} \\ \tau_{xz} &= \frac{E}{2(1+\nu)} \gamma_{xz} \end{aligned} \quad (1.80)$$

This is more commonly represented in Voigt's notation (keeping in mind that

stresses and strains are not truly vectors).

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{pmatrix} \quad (1.81)$$

Equation (1.81) is known as the **generalized Hooke's law**. It is the full, 3D constitutive theory representing linear elastic behavior of solids. The classical 1D Hooke's law is a special case of this theory; if all stresses except for σ_{xx} are set to zero, then equation (1.81) reduces to (1.69).

1.2.3.3 Strain Energy

If a mass is displaced by a force, parallel to that force, work is done on the mass. Work is a form of mechanical energy. Since stress is a measure of internal force and strain is a measure of displacement, together they logically produce some form of work. In the case of stresses and strains though, an incremental change in strain is associated with a given stress, as illustrated in Figure 1.28. The work of deformation for each increment, denoted Δu , is stress times the deformation of that increment:

$$\Delta u = \sigma \cdot \Delta \varepsilon \quad (1.82)$$

To get the total work, simply sum each incremental term. Since the stress is

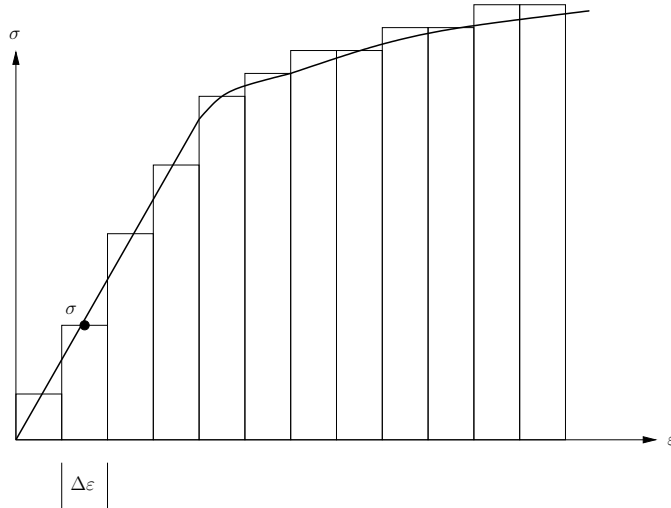


Figure 1.28: Incremental loading

continuous, the increments become differential and the sum becomes integration:

$$u = \int \sigma d\varepsilon \quad (1.83)$$

The units of u are the same as the units of stress, but instead of force per area they may be interpreted as *energy per volume*. Equation (1.83) defines the **strain energy density** u (sometimes denoted w or π). Stress and strain, and therefore the strain energy density, are defined pointwise in the continuum. The total strain energy may be determined by summing u for every particle in the body; that is, integrating the strain energy density over the volume of the object:

$$U = \int u dV \quad (1.84)$$

The strain energy density is generally more useful and more informative than the total strain energy. From the definition of u in equation (1.83) as the integral of the stress with respect to strain, it is clear that the strain energy density is represented as the area under the stress-strain curve. The total area under the elastic region is called the **modulus of resilience**, u_r . It represents the maximum amount of energy the material can absorb before taking damage (permanently deforming). The total area of the entire stress-strain curve is called the **modulus of toughness** u_t and represents the total energy the material absorbs before it fractures. These properties may be important considerations in applications such as blast mitigation and impact safety.

Consider a material loaded into the elastic region. It has some strain energy shown as the area under the curve. When this material is unloaded, it unloads along the same path, recovering all of the energy. Elasticity stores energy, much like a spring. If the material is loaded into the plastic region, it also has an associated strain energy shown as the area under the curve. When it is unloaded, it follows a path parallel to the elastic curve. The recovered energy is the area under the unload curve, shown shaded in Figure 1.29. The rest of the strain energy is not stored in the material, or it would still behave like a spring. Instead, it is lost, dissipated as some non-recoverable (typically thermal) energy. This energy dissipation is directly related to the permanent deformation of plasticity; following dissipation, there is no energy to convert to the necessary work for full recovery of the deformation.

One additional important observation is that Poisson effects do not result in any strain energy. The forces associated with Poisson effects are perpendicular to the deformation, so the work generated is zero.

1.2.3.4 Ductility

Closely related to plasticity is the concept of **ductility**. Ductility is the amount of plastic deformation a material can undergo before fracture. A material with low ductility is called **brittle**. There are two primary measures of

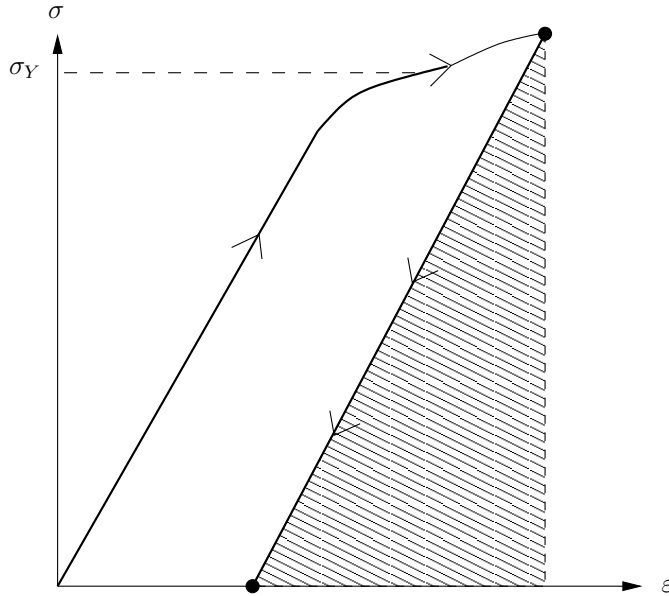


Figure 1.29: Energy recovery and dissipation in plastic deformation

ductility, **percent elongation** ($\%EL$) and **percent area reduction** ($\%AR$).

$$\%EL = \frac{L_f - L_0}{L_0} \times 100 \quad (1.85)$$

$$\%AR = \frac{A_0 - A_f}{A_0} \times 100 \quad (1.86)$$

Note that by equation (1.85), percent elongation is equivalent to the engineering strain at failure.

Brittle is often taken to mean the same thing as *hard*, but the two are quite different. **Hardness** is a measure of the strength of a material, its resistance to localized damage such as scratches or indentations, and may related to the yield stress. Brittleness is related to the deformability of a material. Consider the stress-strain curves shown in Figure 1.30.

A material may have high strength, but low ductility, or vice versa. Alternatively a material may be both strong and ductile, or weak and brittle. The two properties are not necessarily equivalent. A strong but brittle material may require high stresses to yield, but failure occurs shortly thereafter. On the other hand, a weaker but more ductile material will have a lower but potentially much wider stress-strain curve. As the ductile curve tends to cover a larger area, ductile materials in general can absorb much greater amounts of strain energy than brittle materials. Even a much stronger brittle material will absorb far less energy because the elastic region of the stress-strain curve is so small for

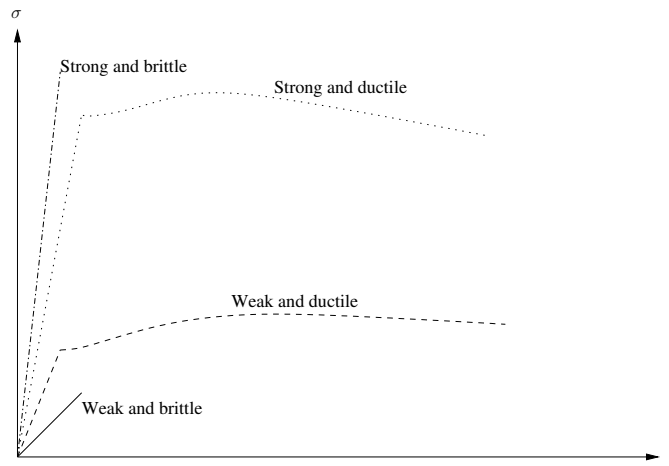


Figure 1.30: Ductility and strength

most materials, as shown in Figure 1.31 for 1018 steel, 2024 aluminum, and 360 brass.

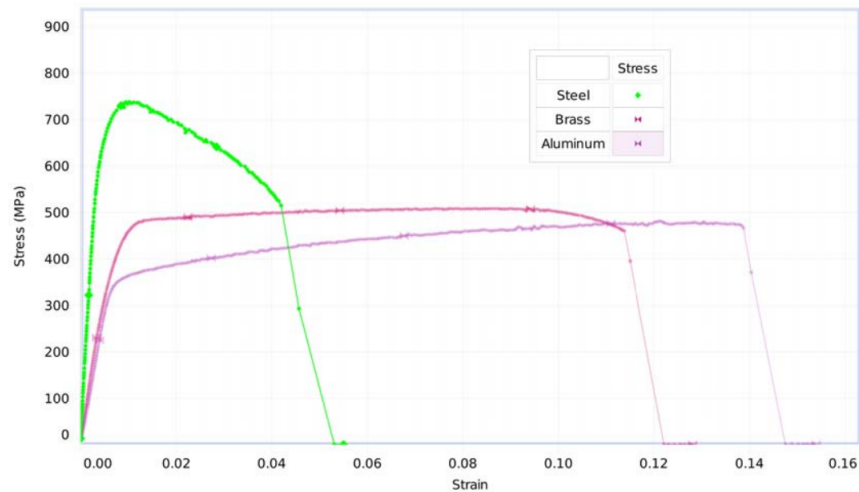


Figure 1.31: To-scale stress-strain curves for various metals

Consider Figure 1.24 again. Necking is an artifact of ductility and the Poisson effect. A brittle material cannot deform enough for visible necking to occur. In fact, fracture of ductile materials in tensile tests occurs at the neck and clearly shows the plastic deformation with a conical cup around the necking point. Brittle materials on the other hand fail suddenly, with relatively flat, smooth surfaces on the crack face and no visible reduction of area, as shown with the brittle aluminum in Figure 1.32.



Figure 1.32: Brittle failure of an aluminum sample

While not directly dependent on one another, ductility and strength can affect each other. Look at Figure 1.27 again. When initially loaded, the yield stress is between points **A** and **B**. If the material is deformed to point **B** and unloaded, then loaded again to the original yield stress, it will still be elastic. If point **C** is taken as the new reference state with zero strain, the the yield stress of the material will be at point **B**. By plastically deforming the body, the yield strength is increased. However, because some plastic deformation has already occurred, there is less remaining before failure. We have effectively traded ductility for strength. This process is known as **strain hardening**.

Ductility and yield strength may also both be affected by other factors. With increasing rate of deformation (strain-rate), the yield strength of many materials will increase, and ductility will decrease (Figure 1.33(a)). That is, in sudden disturbances such as impacts or blasts, even normally ductile materials may only be able to absorb a fraction of the energy that they can in the quasi-static case. Additionally, as temperature increases, strength is reduced and ductility increased, particularly in metals (Figure 1.33(b)). If the temperature is reduced, there may be some increase in strength, but at a certain point the ductility will rapidly decrease. This is the **ductile-to-brittle transition temperature**. Note that the curves shown in Figure 1.33 are general; not all materials will behave in this way or to such an exaggerated extent.

1.2.3.5 Directional Properties

Recall the stress tensor defined by equation (1.59). The subscripts define *where* and in which direction the stresses act. These are related to the strains through the constitutive theory. However, there is no requirement that all the stresses must be related to the strains in the same manner. In many materials,

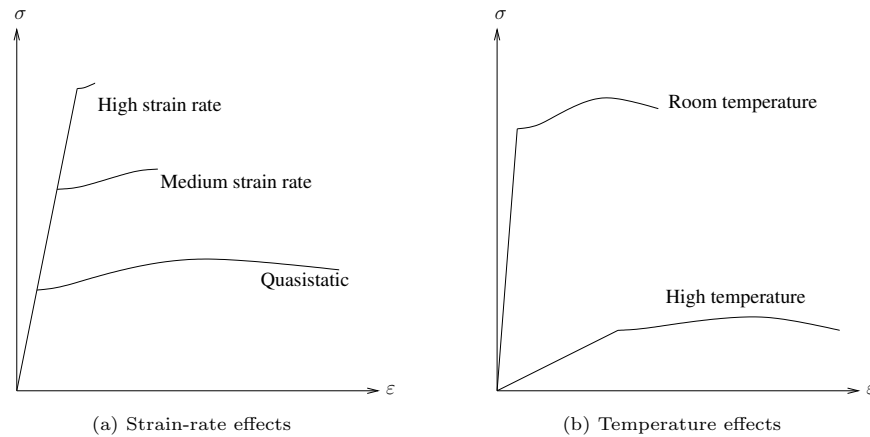
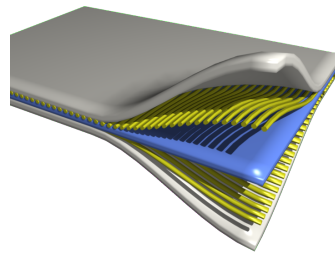


Figure 1.33: External influences on strength and ductility

the constitutive properties are uniform throughout the material and independent of direction. Such a material is called **isotropic**; most metals behave in this way. However, many materials may be stiffer or stronger in certain directions. A good example of this is wood, where the behavior of the material depends entirely on whether it is loaded with the grain or across it (Figure 1.34(a)). Such materials in general are **anisotropic**. Composites and monocrystalline metals are both commonly anisotropic. A special case of anisotropy is **orthotropy**, in which material properties are aligned with the three axes of the coordinate system. Laminated composites formed by layering fiber sheets at different angles are very often orthotropic (Figure 1.34(b)).



(a) Wood grain ([Leonardo Aguiar, Wikimedia Commons, Creative Commons Attribution 2.0](#))



(b) Laminated composite schematic ([PerOX, Wikimedia Commons Public Domain](#))

Figure 1.34: Anisotropic materials

A common source of anisotropy in materials is **heterogeneity**. If the material varies in space, then the properties will vary accordingly. Fibers are often much stiffer along their lengths than in the transverse direction. Embedding fibers in a material will introduce that property into the bulk. Not all heteroge-

neous materials are anisotropic, though. Concrete is an agglomerate of mortar, sand, pebbles, etc., but is often considered isotropic and **homogeneous** for the purposes of mechanical analysis. This is a matter of *scale*: near the interface of the mortar and a pebble, the material behavior will be locally direction dependent and possibly erratic. When the bulk concrete is viewed, however, the total effect of the local anisotropies and heterogeneities are smeared into a homogeneous and isotropic result.

Mathematically, anisotropy will appear most clearly in the constitutive theory. Even a simple linear elasticity model will be affected. The Young's modulus could be direction dependent, giving values E_x , E_y , and E_z rather than the single value E seen in the isotropic case. Poisson's ratio and the shear modulus may be even more complex, with both the containing plane and orientation direction affecting the results. Heterogeneity in turn will cause properties to vary in space; e.g. $E = E(x)$.

Even isotropic materials can have directional dependencies, or at least *sign* dependencies. Many materials behave very differently in compression and tension. The Young's modulus will not change in the case of linear elasticity, but brittle materials especially tend to be much stronger in compression (Figure 1.35). There are several physical phenomena that contribute to this; crack closure, reversed Poisson effects increasing the area, etc.

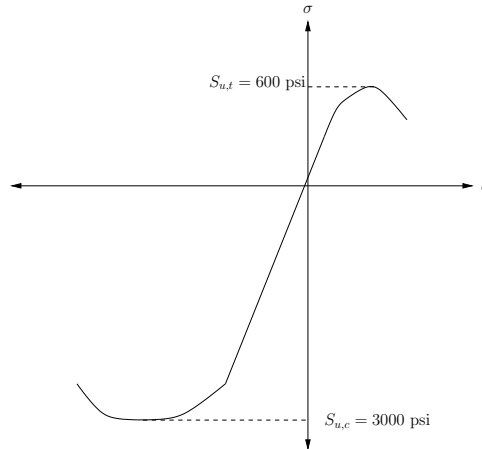


Figure 1.35: Stress-strain diagram for compression and tension of a typical concrete

1.2.3.6 Thermal Strains

Most materials when subjected to an increasing temperature will increase their volume. The ideal gas law is perhaps the most obvious representation of this effect. Gases are not the only materials that experience this phenomenon; both liquids and solids are also subject to **thermal expansion**, albeit somewhat differently. In a solid, this is described by the **coefficient of thermal**

expansion:

$$\alpha_V = \frac{1}{V} \frac{dV}{dT} \quad (1.87)$$

where V is the volume and T is the temperature. The coefficient of thermal expansion for most materials is a function of temperature, however for relatively small temperature changes it can be assumed constant. Furthermore, for small changes in volume, $\frac{dV}{dT} \approx \frac{\Delta V}{\Delta T}$, the total change in volume for the total temperature change. In this case, the thermal expansion can be described by:

$$\Delta V = \alpha_V V \Delta T \quad (1.88)$$

Looking at the linear (one-dimensional) case, a similar relationship holds with regard to length, although the coefficient is not the same.

$$\alpha_L = \frac{1}{L} \frac{dL}{dT} \quad (1.89)$$

Following similar logic as the volumetric case, this can be simplified to:

$$\frac{\Delta L}{L} = \alpha_L \Delta T \quad (1.90)$$

But $\Delta L/L$ is the (engineering) normal strain. Substituting this definition into equation (1.90) and dropping the subscript, we define **thermal strains** as:

$$\varepsilon_{thermal} = \alpha \Delta T \quad (1.91)$$

Thermal strains are related to the volumetric expansion. By determining $\varepsilon_{thermal}$ in each direction, the total change of dimensions can be determined, and from that, the change in volume. It can be shown that for an isotropic material undergoing small deformations over small changes in temperature, the volumetric coefficient of thermal expansion is related to the linear coefficient by a factor of three.

$$\alpha_V = 3\alpha_L = 3\alpha \quad (1.92)$$

Note that thermal strains occur in the absence of forces. They are purely due to the change in temperature. Because of that, thermal expansion does not naturally result in stresses. However, there are two primary means by which so-called **thermal stresses** arise:

1. The temperature change or the material properties are not uniform throughout the body. In this case, the presence of a temperature gradient or heterogeneities will cause different parts of the body to expand at different rates. Regions expanding more will ‘tug’ on neighboring points that are expanding less. This tugging action takes the form of internal forces, i.e., stresses.
2. If the body is constrained and not allowed to fully expand, then the constraints introduce forces to overcome the thermal deformation. Since this expansion process is no longer force-free, stresses develop.

A few materials will actually decrease volume with increasing temperature. This **thermal contraction** is defined exactly the same way as thermal expansion, with a negative coefficient. Zirconium tungstate is a well-studied example. Water will also contract with increasing temperature near the freezing point, between 0 and 5°C.

Example 1.5 (Thermal Expansion).

PROBLEM STATEMENT:

Consider a solid rectangular prism of aluminum. Compute the change of volume using thermal strains, and compare with the volume change determined directly from volumetric expansion.

KNOWN:

- Box dimensions: $5 \times 10 \times 20$ cm
- From Appendix B:
 - $\alpha = 23 \times 10^{-6} \text{ K}^{-1}$
 - $\alpha_V = 69 \times 10^{-6} \text{ K}^{-1}$
- $\Delta T = 20 \text{ K}$

ASSUMPTIONS:

- Small ΔT
- Box is uniformly heated

ANALYSIS:

The change in dimension in each direction can be determined using equation (1.90). Substituting in values of length, width, and height:

$$\begin{aligned}\Delta L &= L\alpha\Delta T = 9.2\Delta T \times 10^{-3} \text{ cm} \\ \Delta W &= W\alpha\Delta T = 4.6\Delta T \times 10^{-3} \text{ cm} \\ \Delta H &= H\alpha\Delta T = 2.3\Delta T \times 10^{-3} \text{ cm}\end{aligned}\tag{1.93}$$

The volume change of the box is computed from the final and initial volumes:

$$\Delta V = V_f - V_0 = (L + \Delta L)(W + \Delta W)(H + \Delta H) - LWH = 1.381 \text{ cm}^3 \tag{1.94}$$

Using equation (1.88) to determine volume change directly:

$$\Delta V = (69 \times 10^{-6} \text{ K}^{-1})(20 \text{ K})(1000 \text{ cm}^3) = 1.380 \text{ cm}^3 \quad (1.95)$$

The two values of ΔV computed in equations (1.94) and (1.95) match almost perfectly. The small difference in their values comes from the assumption of small deformation required to derive equation (1.92). With that assumption, products of ΔL , ΔW , etc. are neglected. The numerical error introduced with this is very minor, as long as the values of ΔL , ΔW , and ΔH are small.

1.3 Analysis and Surrogate Models

Consider a constitutive theory for 3D stress and deformation, expressed using Voigt's notation:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (1.96)$$

There are six stresses, six strains, and $[D]$ is a 6×6 matrix of material relations. At the onset of analysis, little is known about either the deformation or the force fields over the entire domain. Even introducing additional relations, such as force balances or the displacement-strain relations, there are many unknowns representing detailed behavior over potentially complex shapes. This problem is generally completely intractable without advanced simulation tools like finite elements.

The model must therefore be simplified before it can be used. When certain conditions are met and assumptions made, **surrogate models** may replace the complete model. Such models typically reduce the dimension (to 1D or 2D) and neglect some physics. In appropriate circumstances, these surrogates can be quite accurate representations of the physics. In some cases, we accept certain flaws and inconsistencies because of the usefulness of the model. However, great care must be taken to ensure that the surrogate model is suitable for the physical problem; any assumptions made to simplify the analysis must be valid. The remainder of the text will focus on several surrogate models, their development, validity, and applications.

Each specific surrogate model makes use of a different set of assumptions, but there are several assumptions that will hold universally through the remainder of this text, unless specifically noted otherwise:

- The material is homogeneous
- The material is isotropic

- The material is a continuum
- The body is loaded quasistatically
- The material undergoes only small deformation
- The material is linearly elastic
- End-effects and other non-uniformities will be neglected

1.3.1 A Note on Applied Forces

Very commonly in applied mechanics problems, the force is treated as a single vector acting at a location. This is perhaps the most common modeling choice made in practice, but like all models it requires care. Consider a point force acting on a material. This force has a finite magnitude. However, if it is truly applied to a point, then it acts on an infinitesimal area. The average stress on that location is then given by:

$$\sigma = \lim_{A \rightarrow 0} \frac{P}{A} = \infty \quad (1.97)$$

A point force would produce infinite stress. This is known as a **singularity**. No material can support a point force, and in fact they do not exist. As non-physical as they are, however, point loads are very useful modeling tools. Often they are used to illustrate the total magnitude of a force acting over a given area; it may be simpler analytically and computationally to represent a distributed load over a small region as a point load, and simply ignore the local effects of the singularity through Saint-Venant's principle.

In some cases, use of the point force model can create issues. For example, in detailed 3D analysis, local effects of the singularity show very high stresses around the load. Attempts to resolve this issue with more accurate computations will in fact increase the effect. Furthermore, the presence of a force at a single point may lead to discontinuities in the mechanical analysis that are difficult to reconcile with the physics. In circumstances like these, it is perhaps better to consider a physical distributed load (even one acting over a small area) than the simpler, but non-physical, point load.

1.4 Differential Models*

1.4.1 Normal Strain

Let us revisit strain as defined by equation (1.56). In this definition, ds represents the length of a material fiber, or the distance separating a single pair of material points. For the sake of simplicity with discussion, assume that this fiber is aligned with the x -axis, so that $ds = dx$. If a load is applied to this fiber (once again, parallel to the x -axis for simplicity) it will deform to a new length, dx' , as shown in Figure 1.36.

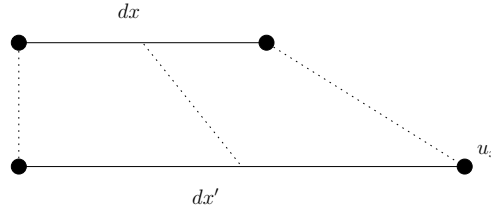


Figure 1.36: Deformation of a single material fiber

Assuming the point on the left is fixed in place, it is clear that each point in the fiber further to the right will displace more. The displacement of the right point u_x is equivalent to the change in length of the fiber, while the point in the middle displaces half of that. The motion of every point in the bar relative to its initial location can be expressed as $\frac{\partial u_x}{\partial x}$. The total displacement of the right point is then given by:

$$u_x = \frac{\partial u_x}{\partial x} dx \quad (1.98)$$

With total displacement known, the change of length is also known. Substituting these values into equation (1.56):

$$\varepsilon_{xx} = \frac{dx' - dx}{dx} = \frac{(dx + \frac{\partial u_x}{\partial x} dx) - dx}{dx} = \frac{\partial u_x}{\partial x} \quad (1.99)$$

The normal strain in the x -direction is given by the derivative of the displacement in that direction. This conforms well with the basic understanding that normal strain is a measure of relative motion; if the points showed the same displacement, the fiber would exhibit rigid body motion and the derivative of the displacement would be zero.

Following similar logic, the normal strains in the y - and z -directions are given by:

$$\begin{aligned} \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\ \varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \end{aligned} \quad (1.100)$$

The normal strains in all three directions are clearly defined in terms of particle displacements.

1.4.2 Shear Strain

Recall that unlike normal strain, shear strain must be defined using two material fibers. Consider Figure 1.37, in which two material fibers were initially parallel to the x - and y -axes but following deformation have changed their alignment. Focusing on the bottom fiber, we understand that the relative motion of the fiber along its length can be described by the derivative of the displacement, similarly to the discussion for normal strains. In this case however, the

displacement is perpendicular to the length of the fiber, so the relative motion is given by $\frac{\partial u_y}{\partial x}$.

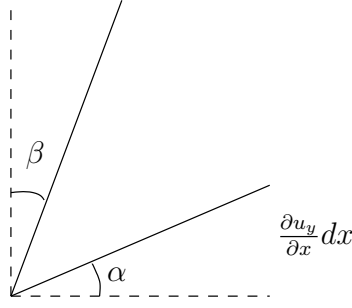


Figure 1.37: Orientation of two material fibers

The total displacement of the rightmost point of the fiber can then be found using the same approach as equation (1.98).

$$u_y = \frac{\partial u_y}{\partial x} dx \quad (1.101)$$

As the material fiber rotates counterclockwise, the rightmost point will move up by u_y , but it will also move left some u_x found using equation (1.98). Shear strain is defined as the change in angle from initially perpendicular material lines. To solve for α , simple trigonometry can be applied to the lower fiber shown in Figure 1.37. The base of the triangle is given by the initial position of the rightmost point, dx , plus the displacement u_x .

$$\tan(\alpha) = \frac{\frac{\partial u_y}{\partial x} dx}{dx + \frac{\partial u_x}{\partial x} dx} = \frac{\frac{\partial u_y}{\partial x}}{1 + \frac{\partial u_x}{\partial x}} \quad (1.102)$$

With the assumption of small deformation, $\frac{\partial u_x}{\partial x} \ll 1$ so may be neglected. Furthermore, small deformation results in a small angle α , allowing for the standard small angle approximation of $\tan(\alpha) \approx \alpha$.

$$\alpha = \tan(\alpha) = \frac{\frac{\partial u_y}{\partial x}}{1} \quad (1.103)$$

Following a similar process for the upper fiber:

$$\beta = \frac{\partial u_x}{\partial y} \quad (1.104)$$

Finally, the shear strain in the xy -plane may be determined. Since the shear strain is the total change of angle, it is simply the sum of the two angles found.

$$\gamma_{xy} = \alpha + \beta = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \quad (1.105)$$

The same process may be followed in the other two cardinal planes to find the remaining shear strains.

$$\begin{aligned}\gamma_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ \gamma_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\end{aligned}\tag{1.106}$$

There are some important points to consider with these results:

- The normal strain in each direction depends only upon the derivative of the displacement of the material particle, both in that direction.
- The shear strain in each plane also depends on derivatives, but perpendicular to the displacements.
- These definitions hold that strain is linearly related to the displacement in all cases.
- All of these conclusions and relations only hold if the assumption of small deformation (as discussed in Section 1.3) is satisfied. With finite deformation, equations (1.99), (1.100), (1.105), and (1.106) are incomplete (finite strain includes additional nonlinear terms). For this reason, these definitions are often referred to as **infinitesimal strain**.
- The definition of shear strain in (1.105) and (1.106) support the previous discussion of symmetry of the shear strains and stresses. If we consider γ_{yx} , the roles of the x - and y -coordinates are reversed.

$$\gamma_{yx} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\tag{1.107}$$

This is mirrored from γ_{xy} , but since addition is commutative it results in exactly the same expression as γ_{xy} .

- The derivations presented in these sections have been significantly simplified to emphasize major points. More complete explanations and derivations may be found in reference materials for continuum mechanics or theory of elasticity.
- Equations (1.99), (1.100), (1.105), and (1.106) together represent the **strain-displacement** relations for deformable media. These are sometimes also called the **kinematic relations**.

1.4.3 Equilibrium

Consider a slender, straight object loaded along its length, as shown in Figure 1.38(a). P_1 and P_2 are surface tractions (modelled as point loads) on either end of this structure, and $F^b(x)$ is a body force (a load distributed over the entire volume, such as gravity).

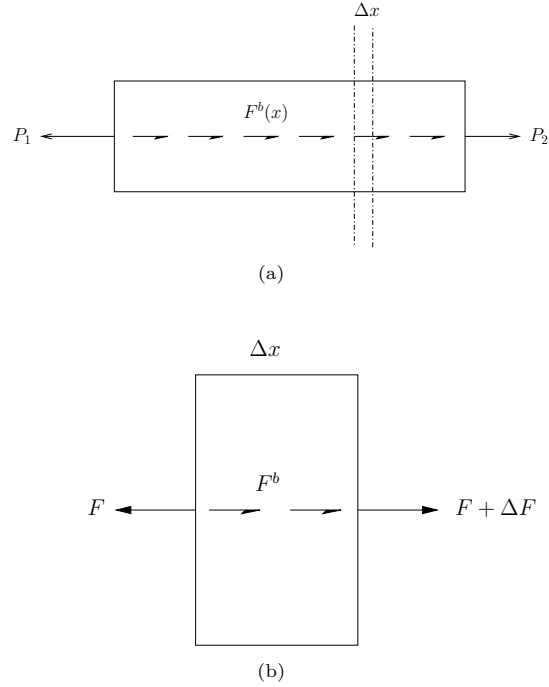


Figure 1.38: One-dimensional loading on a structure

The forces on this structure are such that it is static. Taking a small section of the material Δx , as shown in Figure 1.38(b), the internal forces of the body can be found. Note that these forces are not the same as the externally applied loads P_1 and P_2 and may vary with position x . Due to the presence of the body force, the internal forces on the left and right ends of the segment are not equivalent and differ by some ΔF . The entire structure is static, so the section Δx must be as well. Considering static equilibrium for the schematic shown in Figure 1.38(b):

$$\sum F_x = 0 = (F^b \Delta x) + (F + \Delta F) - F \quad (1.108)$$

which simplifies to:

$$F^b + \frac{\Delta F}{\Delta x} = 0 \quad (1.109)$$

The expression in equation (1.109) holds for a segment of finite size. To consider every possible point in the structure, the segment must be reduced to an infinitesimal size.

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \left(F^b + \frac{\Delta F}{\Delta x} \right) \\ = F^b + \frac{dF}{dx} = 0 \end{aligned} \quad (1.110)$$

Equation (1.110) represents static equilibrium of a single point (or in this case, a 2D slice of material “one point thick”). Assuming the initial structure shown in Figure 1.38 is one-dimensional, i.e., loaded purely along its length, then nothing varies in the y - and z -directions. Since this section is already a single point, the definition of stress given by equation (1.57) holds precisely and reduces to:

$$\sigma = \frac{F}{A} \quad (1.111)$$

Substituting (1.111) into (1.110):

$$F^b + \frac{d}{dx}(A\sigma) = 0 \quad (1.112)$$

This can be further modified by applying Hooke’s law and the kinematic relations.

$$F^b + \frac{d}{dx}(AE\varepsilon) = F^b + \frac{d}{dx}\left(AE\frac{du_x}{dx}\right) = 0 \quad (1.113)$$

Equation (1.113) is known as the **one-dimensional equilibrium equation** for deforming solids. It is based entirely on the application of Newton’s laws with material specifications and the kinematic relations. The equilibrium equation is a second-order differential equation, and is linear as long as the material is linearly elastic.

Following similar, albeit more involved, processes it is possible to construct a complete three-dimensional model for deforming solids. Once again utilizing the assumptions and restrictions discussed in Section 1.3, the **3D equilibrium equations** for deforming solids are given by:

$$\begin{aligned} \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + F_x^b &= 0 \\ \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} + F_y^b &= 0 \\ \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + F_z^b &= 0 \end{aligned} \quad (1.114)$$

This system of partial differential equations represents static equilibrium in a deformable continuum. By substituting the generalized Hooke’s law (1.81) and kinematic relations (1.99), (1.100), (1.105), and (1.106), equations (1.114) simplify to a system second-order, linear, partial differential equations in displacements u_x , u_y , and u_z .

- The solution of these equations fully describe the displacement of every point in a continuum.
 - Note that displacements alone do not represent deformation. However, if displacements are known, then it is trivial to use the kinematic relations to determine the strain (i.e., the displacement of each point relative to its neighboring material particles).

- Continuing this process, the stress is deterministic from the strain through Hooke's law.
- In summary, equations (1.114) provide a mechanism for a complete solution of any mechanics problem satisfying the basic assumptions.
- While broadly descriptive, the 3D model is not immediately solvable. Solutions of partial differential equations are not trivial and strongly depend on geometry and boundary conditions, both of which may be quite complicated in practical application.
- This model is most often used in conjunction with computational tools to remediate the difficulties present in generating analytical solutions. Perhaps the most common of these tools in modern engineering practice is the **finite element method**. While very often used for solid mechanics, the finite element method is in fact a mathematical tool for approximating solutions of differential equations in general, and so is not discussed in detail here.

Chapter 2

Bar Analysis

The first surrogate model considered is the **bar**. Bars are perhaps the simplest structural member, reducing to a completely one-dimensional model. For all their relative simplicity, bars are still very useful for at least two reasons:

1. They are relatively common in structural applications. Trusses, for example, are composed of individual bars (known as **two-force members** in statics lingo). Generally, it is much more efficient to analyze such structures as collections of bars rather than 3D continua.
2. They provide valuable insight into complex physical phenomena through an easy-to-understand lens. Many lessons learned from the study of bars hold much more broadly.

2.1 Fundamentals of Bar Models

As already mentioned, bars are 1D models. The majority of the restrictions required for bar analysis are therefore geometric. In addition to the basic assumptions listed in Chapter 1, we make the following additional assumptions:

1. The structure is loaded only by forces oriented along its longitudinal axis
2. The forces are approximately centered on the axis of the member
3. The structure is long, slender, and straight
4. Area changes occur gradually

Each of these assumptions, while seemingly straightforward, are profoundly important to simplifying the physical model. We begin discussion with the first assumption. If our goal is to reduce the model to one dimension, this assumption is perhaps obvious. We cannot consider any off-axis loads without introducing deformation in other directions. These axial loads take the form of an x -force (for example) distributed across a cross-section in the yz -plane. When paired

with the second assumption, the total force of that axial load may be represented as a point force acting on the centerline of the bar, as shown in Figure 2.1.



Figure 2.1: Force behavior in bars

More of a physical restriction than an assumption, the condition that the member must be long, slender, and straight somewhat naturally follows from the first two assumptions. In order for a two-force member to be static, the forces must align. This will only happen for all forces on a body if the body is straight. If it is long and slender, there is another advantage. Consider the bar shown in Figure 2.2. An arbitrary, approximately symmetric load is applied at the end of the bar. In the vicinity of this load, the internal forces reflect this behavior. However, as distance from the applied load increases, the particles drag on one another and the distribution smooths out. At some distance from the disturbance, the internal force field has become uniform. Following the discussion in Section 1.2.2, every particle in the cross-section carries the same load. The stress in the bar is therefore equivalent to the average stress:



Figure 2.2: Internal force behavior in bars

$$\sigma_{xx} = \frac{N}{A} \quad (2.1)$$

Naturally, Saint-Venant's principle must be applied near disturbances such as the applied load. While there is no hard rule, a distance equal to the larger of the cross-sectional dimensions is typically considered sufficient. Since the bar is long and slender (the axial dimension is much greater than the transverse), this is a relatively localized effect. For the purposes of work in this text, non-uniformities are typically assumed negligible.

The final assumption for bar analysis is important to maintain a 1D problem. If the area changes rapidly, as shown in Figure 2.3(a), then the internal forces are not aligned with the x -axis. The cross-sectional area of the bar may change along its length, but it must do so gradually. In an interesting juxtaposition, a sudden area change is actually more appropriate for bar models. As shown in Figure 2.3(b), there are non-uniformities in the internal force near the area change (especially on the larger side), but these fade with distance according to Saint-Venant's principle.

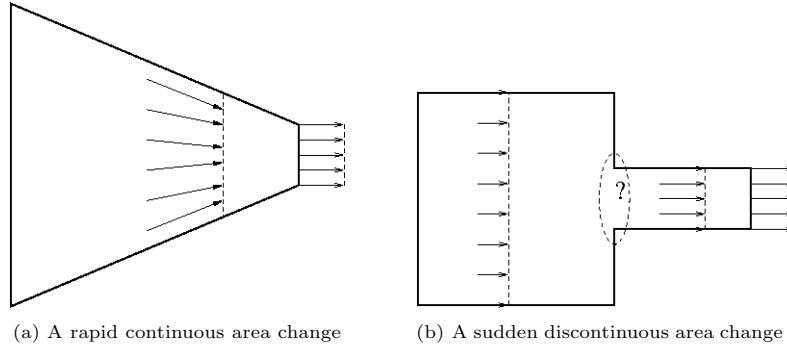


Figure 2.3: Effects of variable area on internal forces

With these assumptions, the free-body diagram of a bar reduces to its centerline. Forces distributed across the cross-section are modeled as point loads, the area becomes a parameter, and only the x -coordinate is relevant. With this in mind, it is clear why bar problems are also often called **uniaxial members**.

2.2 Deformation of Bars

Consider the bar of length L shown in Figure 2.4(a). The bar is subjected to point loads P_1 and P_2 (units of force). $F(x)$ is a load distributed *along the length* (units of force per length). The area $A(x)$ is exaggerated for visibility. Due to the applied loads, the length of this bar changes by some δ .

We want a general analysis, so we consider a small slice of the bar Δx (Figure 2.4(b)). On either side of this slice are internal normal forces N_1 and N_2 . The distributed load $F(x)$ acts over this slice. If the entire body is deformed due to the applied loads, then this slice must deform due to the internal forces acting on it, denoted as $\Delta\delta$. The *average* strain in this slice is defined as:

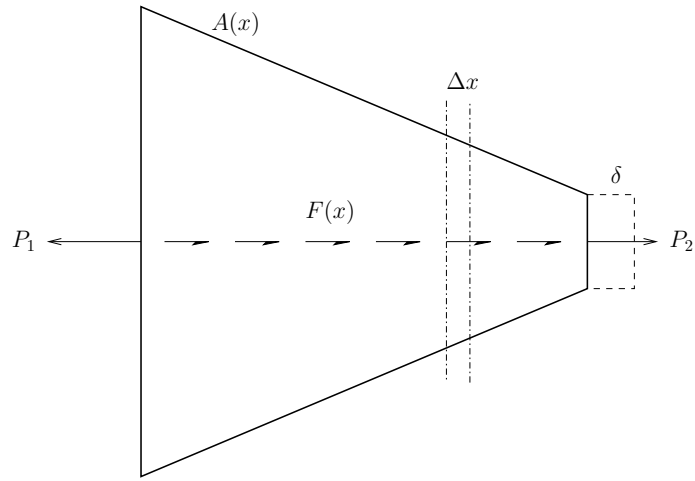
$$\varepsilon_{xx} = \frac{\Delta\delta}{\Delta x} \quad (2.2)$$

Substituting equations (2.1) and (2.2) into Hooke's law (1.69):

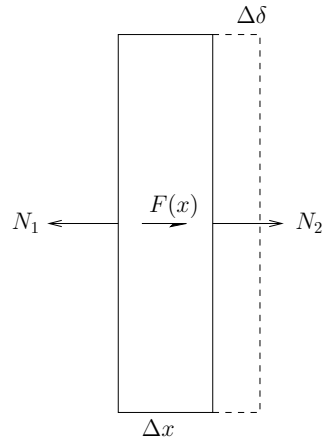
$$\frac{N}{A} = E \frac{\Delta\delta}{\Delta x} \quad (2.3)$$

Since the slice is so small, the area is approximately constant for each slice or location. We are interested in the behavior of the bar at every point, therefore our slice Δx must be taken infinitesimally small, differential. In this case, the width of the slice becomes dx and the difference between N_1 and N_2 vanishes.

$$\frac{N(x)}{A(x)} = E \frac{d\delta}{dx} \quad (2.4)$$



(a) A generic bar problem



(b) A small internal section of the bar

Figure 2.4: Deformation behavior of a bar

The normal force and area both vary with location. Solving this equation algebraically:

$$d\delta = \frac{N(x)}{EA(x)} dx \quad (2.5)$$

Equation (2.5) is the local deformation of the bar at a point. To get the total macroscopic change of length of the bar, we must sum up each point. That is, we integrate.

$$\delta = \int_L \frac{N(x)}{EA(x)} dx \quad (2.6)$$

In equation (2.6), L is the length of the bar (or segment of the bar) and

$A(x)$ is the area, both straightforward geometric parameters. E is the Young's modulus, a constant due to the assumption of homogeneity. That leaves only determination of the internal normal force $N(x)$ to complete the analysis.

2.2.1 Internal Forces

As mentioned in Section 1.1.4, internal forces at any location must balance with the external forces to maintain static equilibrium throughout the body. In the case of bars, the only external loads are axial forces. Thus, the internal forces are all axial as well. To determine the internal forces, take a *section* in the body. The internal forces act on the exposed face of the section, and must balance with the external forces *on either side of the section*. Consider the bar shown in Figure 2.5(a) (recall that the free-body diagram for a bar reduces to a single line). Section $a - a$ is taken perpendicular to the bar. The internal view at the section in Figure 2.5(b) shows the internal force on the face of the section. Since there are only axial forces, the internal force is perpendicular to the section and is called the **internal normal force**. Normal force vectors are by convention positive outward from the section. By Newton's third law, the internal normal forces on either side of a section must be equal in magnitude but opposite in direction.

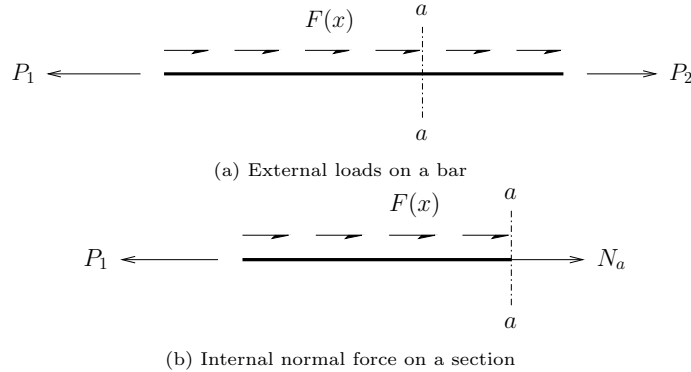


Figure 2.5: Determination of internal normal forces in a bar

Note that the internal normal force \mathbf{N} is a vector, but we typically treat it as a signed scalar value N as in equations (2.4) – (2.6). In other words, we are considering only the normal component of the total internal force vector (which is not necessarily the same as the x -component of the force vector). The internal normal forces are determined by applying force equilibrium to the body on either side of the section (the solution for N will be the same regardless of which side of the section is considered). If the external loads do not change between two points on a bar, then the internal loads will not either. Therefore, it is only necessary to take sections wherever the external loading changes. Some care is still required. Sections should never be taken exactly at a point load, but to either side of one. Distributed loads require additional attention.

Example 2.1 (Internal Forces).

PROBLEM STATEMENT:

Determine the internal forces in the bar shown in Figure 2.6(a).

KNOWNs:

- Point loads are applied at locations Q, R, and S

ANALYSIS:

To determine the internal forces, sections must be taken every time the external loading changes. Therefore, consider section $a - a$ between Q and R and section $b - b$ between R and S.

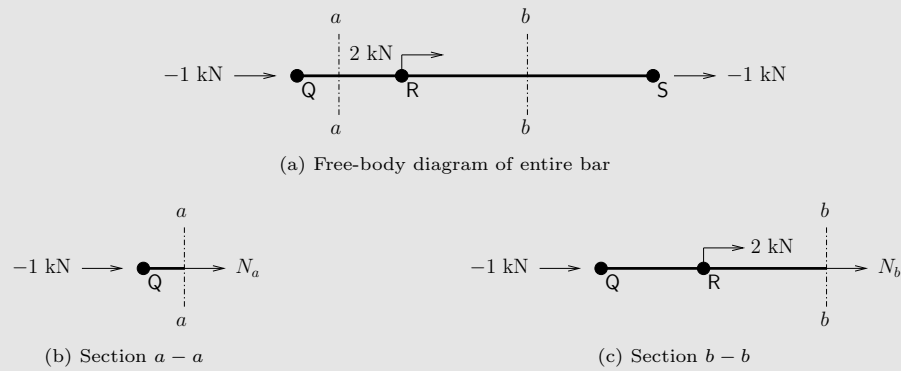


Figure 2.6: Example 2.1

Apply equilibrium at each section to determine the value of the internal normal force at that location.

Section $a - a$:

$$\sum F_x = 0 = -1 + N_a \quad \implies \quad N_a = 1 \text{ kN} \quad (2.7)$$

Section $b - b$:

$$\sum F_x = 0 = -1 + 2 + N_b \quad \implies \quad N_b = -1 \text{ kN} \quad (2.8)$$

Because the external loading only changes at points Q, R, and S, the internal normal force is constant between them. Therefore, N_a describes the internal

normal force at every location between Q and R, and N_b is the internal normal force at every point between R and S. Thus, the internal normal force can be expressed as a piecewise function:

$$N(x) = \begin{cases} 1 \text{ kN} & \text{Q} < x < \text{R} \\ -1 \text{ kN} & \text{R} < x < \text{S} \end{cases} \quad (2.9)$$

Note the sign change in N_a and N_b . Positive internal normal forces are tensile, whereas negative internal normal forces are compressive. The force field in this bar transitions from tension to compression at point R. The left part of the bar is being pulled apart while the right part of the bar is squeezed.

Example 2.2 (Internal Forces).

PROBLEM STATEMENT:

Consider a prismatic steel bar with cross-sectional area $A = 0.5 \text{ m}^2$ and length $L = 3 \text{ m}$, hanging from one end. Determine the internal force field.

KNOWNs:

- $A = 0.5 \text{ m}^2$
- $L = 3 \text{ m}$
- From Appendix B:
 - Density $\rho = 7850 \text{ kg/m}^3$

ANALYSIS:

The bar is suspended from one end, so the weight of the bar aligns with its axis (oriented positive downward from the support), as shown in Figure 2.7(a). To determine the internal forces, we must first determine the external forces.

The weight is not a single force; every particle in the bar is subjected to its own weight. This translates to a distributed load over the volume:

$$W = \rho g = 77,010 \text{ N/m}^3 \quad (2.10)$$

where $\rho = 7850 \text{ kg/m}^3$ is the density of the steel and g is gravitational acceleration. Since this is a bar, the weight is uniformly distributed over each cross-section and varies only with the length.

$$F(x) = WA(x) = (77010 \text{ N/m}^3) (0.5 \text{ m}^2) = 38.5 \text{ kN/m} \quad (2.11)$$

Thus, the weight force translates to a uniformly distributed load for this problem. In addition to the weight, the external reaction at the support must also be found. Consider the free-body diagram shown in Figure 2.7(b) and take equilibrium:

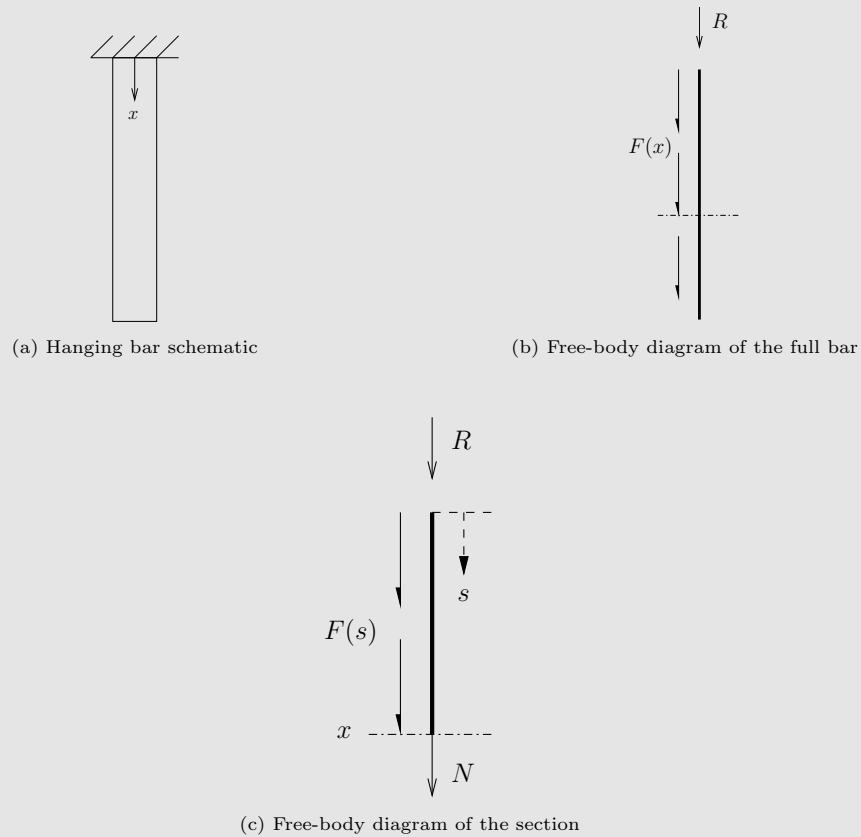


Figure 2.7: Example 2.2

$$\begin{aligned}
 \sum F_x = 0 &= R + \int_L F(x) dx \\
 &= R + FL \\
 &= R + (38.5 \text{ kN/m})(3 \text{ m}) \quad \implies \quad R = -115.5 \text{ kN}
 \end{aligned}
 \tag{2.12}$$

Since $F(x)$ is a constant, the integration reduces to a simple product.

With all external loads known, it is now possible to find the internal forces. Recall that sections are required everywhere the load changes. In this case,

there is a continuous distributed load; that is, every single point in the bar has a force applied to it. By this logic, we now require infinitely many sections. Fortunately for us, there is a convenient mathematical tool for precisely this situation.

Take a section at any arbitrary location x inside the distributed load and draw a free-body diagram of the section, as shown in Figure 2.7(c). Define a coordinate s that describes the location of the section. The point x is arbitrary, so the section's location may be taken at any point in the distributed load. While looking at a section, the distributed load is a function of s rather than x (x has become a single point). By defining the origin of the coordinates at the same point, x is directly replaced by s without a new functional representation required. By allowing the coordinate s to go to the variable x , we effectively collapse infinitely many sections into a single term.

To determine the internal force, take equilibrium of the section. However, keep in mind that determination of the force due to the distributed load requires a length term. If the section does not have a set location, how is that term deterministic? The answer takes advantage of our variable s . By allowing s to go to any distance x , we can translate the variable location of the section into a well-defined function of our axial coordinate.

$$F_{total} = \int_{L_{section}} F(s) ds = \int_0^x F(s) ds = 38.5x \text{ kN} \quad (2.13)$$

By integrating up to x in equation (2.13), the total force due to the distributed load now depends on where in the bar we are looking. With this, apply equilibrium to the section:

$$\begin{aligned} \sum F_x = 0 &= R + \int_0^x F(s) ds + N \\ &= -115.5 + 38.5x + N \\ \implies N(x) &= 115.5 - 38.5x \text{ kN} \end{aligned} \quad (2.14)$$

The internal normal force is a function of x . Due to the distributed load, $N(x)$ varies continuously along the length of the bar. At the top of the bar $x = 0$, $N = R$ and the material is supporting the entire weight of the bar. Conversely, at the bottom of the bar $x = L$, $N = 0$ because it is not supporting anything. Whenever there is a distributed load, the section coordinate s is required. This coordinate is integrated up to any location x allowing for a continuous description of the internal normal force. Keep in mind that in general, integration is required to utilize distributed loads in the equilibrium equations. The integral only reduced to multiplication in this case because the distributed load had a constant value.

2.2.2 Applications of the bar equation

The bar equation (2.6) describes the elongation of a bar due to the internal force field. Recall the sign convention for internal forces; positive internal normal forces point outward from the section, putting the bar in a state of **tension**. According to equation (2.6), a positive N results in a positive δ , lengthening the bar. Likewise, a negative internal normal force is **compressive** and results in a shortening of the bar. Furthermore, since the stress is defined as internal force divided by the cross-sectional area and the strain is defined as the change of length divided by the original length, they follow the same sign convention. Therefore the signs are all consistent; positive stresses/forces are related to positive strains/deformation.

At this point it is useful to refine equation (2.6) somewhat. As it is expressed, equation (2.6) describes the change of length of the entire bar. However, it can also define the change of length of specific portions of the bar. Consider two points a and b . We can determine the internal normal force and area between those points, and integrate between them to get the deformation of just that portion of a bar.

$$\delta_{b/a} = \int_a^b \frac{N(x)}{EA(x)} dx \quad (2.15)$$

The notation is important for clarity; $\delta_{b/a}$ is the elongation of segment $a - b$, but more precisely it is the motion of point b measured relative to point a . If we switch the order of the points, the integration changes direction:

$$\delta_{a/b} = \int_b^a \frac{N(x)}{EA(x)} dx = - \int_a^b \frac{N(x)}{EA(x)} dx \quad (2.16)$$

In this case we are still looking at the same segment of bar, but the sign has changed. It is not in compression from this point of view; $\delta_{a/b}$ is the motion of point a relative to point b . If b moves to the right from a , then equivalently a moves to the left from the perspective of b .

We will now consider a few important examples to illustrate the use of this relation.

Example 2.3 (Constant Internal Normal Force).

PROBLEM STATEMENT:

Determine the elongation of the bar shown in Figure 2.8(a).

KNOWNs:

- Bar is prismatic

- Single point load P

ASSUMPTIONS:

- The material is linearly elastic
- The load is such that the bar undergoes only small deformation
- Length is much greater than transverse dimensions
- Neglect end effects

ANALYSIS:

Consider a simple bar loaded in tension by a single point force P at either end, as shown in Figure 2.8(a). The cross-sectional area of the bar is constant along its length. The origin of the coordinate system is located at the left end of the bar, with the x -axis oriented along the length. Applying the assumptions listed above, this member is long and slender and all forces on the bar align with the axis. As stated previously, all problems in this text are homogeneous, isotropic, and continuous unless otherwise stated. The material is also assumed to be linearly elastic. *This structure can therefore be modeled as a bar.*

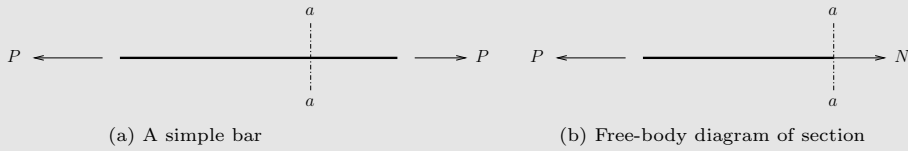


Figure 2.8: Example 2.3

Take section $a - a$ and draw a free-body diagram of the left side (Figure 2.8(b)). The section may be taken at any point between the applied loads. Applying static equilibrium:

$$\sum F_x = 0 = N_a - P \quad \implies \quad N_a = P \quad (2.17)$$

Substitute the normal force into equation (2.6).

$$\delta = \int_0^L \frac{N}{AE} dx = \int_0^L \frac{P}{AE} dx \quad (2.18)$$

P , A , and E are all constants, so the integral reduces to:

$$\delta = \frac{PL}{AE} \quad (2.19)$$

DISCUSSION:

This result holds for any prismatic bar subjected only to point loads at the ends. While only a special case of the general bar equation, this simplified equation is very useful for many applications.

One interesting observation can be made from this result. Rearrange equation (2.19) to isolate the force P :

$$P = \frac{AE}{L} \delta \quad (2.20)$$

Equation (2.20) states that the force is equal to a constant times the change in length, exactly like Hooke's law for linear springs. In fact, Hooke's law for springs is a direct consequence of continuum mechanics. The spring stiffness in this case is:

$$k = \frac{AE}{L} \quad (2.21)$$

k is termed the **extensional stiffness** of the bar, and is a constant when A , E , and L are constants. Somewhat related, the **axial rigidity** is defined as the product AE .

Example 2.4 (Multiple Point Loads).**PROBLEM STATEMENT:**

Determine the internal force the bar shown in Figure 2.9. It is fixed at the left end, with several point loads applied at various locations. The bar is prismatic (constant area), and the coordinate system is anchored at the fixed support.

KNOWNs:

- Bar is prismatic
- Fixed-support at left end

ASSUMPTIONs:

- The material is linearly elastic
- The load is such that the bar undergoes only small deformation
- Length is much greater than transverse dimensions

- Neglect end effects

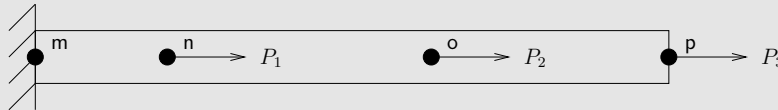
ANALYSIS:

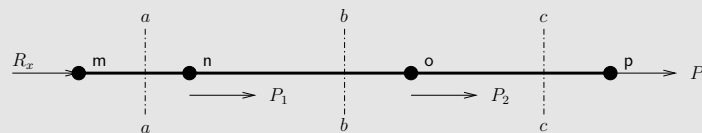
Figure 2.9: Example 2.4

Once again, the first and most critical step is determination of the proper model. Based on the knowns and assumptions, this structure satisfies all requirements for *bar analysis*.

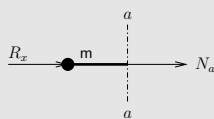
Once determination of the appropriate model is made, the next step in any mechanical analysis (if it's not already done like example 2.3), is to draw the free-body diagram and determine any unknown external reactions. Using the free-body diagram shown in Figure 2.10(a), apply static equilibrium to the bar.

$$\sum F_x = 0 = R_x + P_1 + P_2 + P_3 \quad \implies \quad R_x = -(P_1 + P_2 + P_3) \quad (2.22)$$

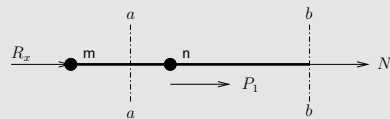
To analyze deformation, we must have internal forces. In this case, the loading changes along the length of the bar, so sections must be taken in each region. Apply static equilibrium at each section.



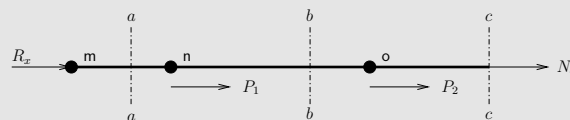
(a) Free-body diagram of entire bar



(b) Section a - a



(c) Section b - b



(d) Section c - c

Figure 2.10: Free-body diagrams

Section a – a:

$$\sum F_x = 0 = R_x + N_a \quad \implies \quad N_a = -R_x = P_1 + P_2 + P_3 \quad (2.23)$$

Section b – b:

$$\sum F_x = 0 = R_x + P_1 + N_b \quad \implies \quad N_b = -R_x - P_1 = P_2 + P_3 \quad (2.24)$$

Section c – c:

$$\sum F_x = 0 = R_x + P_1 + P_2 + N_c \quad \implies \quad N_c = -R_x - P_2 - P_3 = P_3 \quad (2.25)$$

The internal normal force is defined piecewise as:

$$N(x) = \begin{cases} N_a & \text{m} < x < \text{n} \\ N_b & \text{n} < x < \text{o} \\ N_c & \text{o} < x < \text{p} \end{cases} \quad (2.26)$$

To determine the deformation of the bar, we must use (2.26) in (2.6). Since the internal normal force is piecewise defined, the integration must also be performed in a piecewise manner.

$$\begin{aligned} \delta &= \int_{\text{m}}^{\text{p}} \frac{N(x)}{AE} dx \\ &= \int_{\text{m}}^{\text{n}} \frac{N_a}{AE} dx + \int_{\text{n}}^{\text{o}} \frac{N_b}{AE} dx + \int_{\text{o}}^{\text{p}} \frac{N_c}{AE} dx \end{aligned} \quad (2.27)$$

Note that all terms are constant in each segment of the bar. Therefore, the result of equation (2.19) can be applied independently to each segment:

$$\delta = \frac{N_a L_a}{AE} + \frac{N_b L_b}{AE} + \frac{N_c L_c}{AE} \quad (2.28)$$

DISCUSSION:

For any bar with constant properties over each segment of its length, equation (2.19) generalizes to:

$$\delta = \sum \frac{N_i L_i}{A_i E} \quad (2.29)$$

We could also consider this problem in terms of equation (2.16). Notice in Figure 2.9 the points labeled *m*, *n*, *o*, and *p*, corresponding to where each point load is applied. The motion of point *o* relative to point *n*, for example, is found using the properties of the bar in that segment:

$$\delta_{\text{o/n}} = \frac{N_b L_b}{AE} \quad (2.30)$$

The total deformation of the bar is equal to the sum of the deformation in each segment. Expressing that in the relative motion notation:

$$\delta = \delta_{p/m} = \delta_{n/m} + \delta_{o/n} + \delta_{p/o} \quad (2.31)$$

This is equivalent to equation (2.29), but perhaps more informative. The motion of one point relative to another can be constructed by any combination of points between them.

One additional point to note is that although the bar overall is long and slender, the point loads applied at the various locations create local non-uniformities in the stress and deformation fields. While there is sufficient length between point loads to apply Saint-Venant's principle, there will be some inaccuracy in the total deformation and stress values due to these irregularities. Such is the nature of a surrogate model.

Example 2.5 (Distributed Loads).

PROBLEM STATEMENT:

Determine the elongation of the bar shown in Figure 2.11, and quantify the stress field. It is fixed at the left end, with a uniform distributed load applied along its entire length. The bar is circular with constant diameter. The origin of the coordinate system is located at the fixed support.

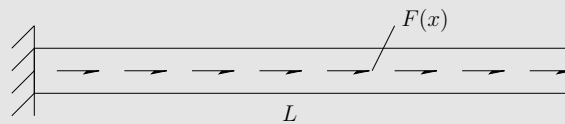


Figure 2.11: Example 2.5

KNOWN:

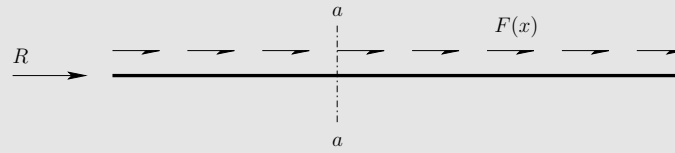
- $L = 1 \text{ m}$
- $D = 0.1 \text{ m} \longrightarrow A = 0.00785 \text{ m}^2$
- $E = 3 \times 10^8 \text{ N/m}^2$
- $F(x) = 2500 \text{ N/m}$

ASSUMPTIONS:

- Neglect end effects

ANALYSIS:

As always, the first step in analysis is ensuring that the problem satisfies the necessary requirements. This rod is long, slender, straight, and loaded only by axial forces. The material is linearly elastic. Upon solving, we will confirm small deformation. Therefore, *bar analysis* is the appropriate approach. Begin with a free-body diagram of the bar and solve for the reaction at the wall (Figure 2.12(a)).



(a) Free-body diagram of entire bar

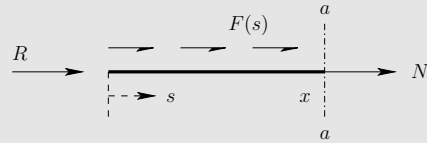
(b) Section $a - a$

Figure 2.12: Free-body diagrams

$$\begin{aligned}
 \sum F_x = 0 &= R + \int_L F(x) dx \\
 &= R + \int_0^1 2500 dx \\
 &= R + 2500 \text{ N} \quad \implies \quad R = -2500 \text{ N}
 \end{aligned} \tag{2.32}$$

To determine the internal normal force, a section must be taken. Following Example 2.2, the section may be taken at an arbitrary location in the distributed load, defined using the sectional variable s (Figure 2.12(b)). The value of s varies from 0 at the left boundary to the location x of the section. Take equilibrium of the forces in this section:

$$\begin{aligned}
 \sum F_x = 0 &= R + \int_{L_s} F(s) ds + N_a = R + \int_0^x 2500 ds + N_a \\
 &= -2500 + 2500x + N_a \quad \implies \quad N_a = 2500 - 2500x \text{ N}
 \end{aligned} \tag{2.33}$$

With the internal force defined, apply equation (2.6) to determine the change of length:

$$\delta = \int_L \frac{N(x)}{AE} dx = \int_0^1 \frac{2500 - 2500x}{(0.00785)(3 \times 10^8)} dx = 0.000531 \text{ m} \quad (2.34)$$

This value of elongation is much less than the length of the bar, thereby satisfying the small deformation requirement.

The stress in the bar is equivalent to the average stress, as per the discussion in Sections 1.2.2 and 2.1.

$$\sigma_{xx} = \frac{N(x)}{A} = \frac{2500 - 2500x}{0.00785} = 318,500(1 - x) \text{ N/m}^2 \quad (2.35)$$

The stress, like the internal normal force, varies along the length of the bar. Note that its peak value occurs at the wall ($x = 0$). Due to the nonuniformities in that region, the value is likely not completely accurate at that point, but we neglect those local effects here. Likewise, the strain may be computed using Hooke's law. It is not equivalent to the average strain (total change of length divided by total length), which would be a constant value.

2.3 Statically Indeterminate Bars

Recall the definition of static indeterminacy from Section 1.1.3. In short, in a statically indeterminate problem, the equations of static equilibrium do not provide enough information to solve for the unknown forces. However, by using our knowledge of the deformation of a bar, it is possible to overcome this hurdle.

How we use the deformation to make a statically determinate problem solvable is critical. Remember, we do not know what δ is without the internal force field, but we cannot solve for the forces without the deformation. The key to this riddle is known as a **compatibility condition**. Compatibility conditions are pieces of information about the deformation field that are known prior to solving the problem. They are typically constructed from boundary conditions or other constraints.

The definition of compatibility conditions is *the* critical step for the evaluation of statically indeterminate problems. Compatibility conditions are not just the constraints, however. They must be expressed in a useful form. Consider the bar constrained at both ends shown in Figure 2.13 (sometimes called a **fixed-fixed bar**). The boundaries at the left and right ends prevent displacement of either of those points. This information is not particularly useful by itself though. At no point does the displacement of a single point come into the equations of static equilibrium or the bar equation. If we are careful, though, we can use the displacement of one point *relative to another*. Since both ends of

the bar are fixed in place, we can state that the motion of the right end relative to the left end is zero, thus our compatibility equation becomes:

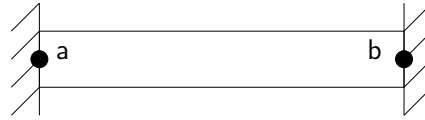


Figure 2.13: A fixed-fixed bar

$$\delta_{b/a} = 0 \quad (2.36)$$

The compatibility condition supplies a numerical value for the relative motion (in this case) that can be used in the bar equation. This allows us to use the bar equation to bring closure to the statically indeterminate problem.

Care must be taken. Compatibility conditions are entirely unique to each problem and require critical thought and evaluation. It will not always be the case that relative motion between two points is zero. There may be varying motions, or lengths of different bars changing together, or even deformation of one member that does not begin until a certain deformation has been reached by another member. The possible variations are limitless, but the compatibility condition is always determined using information about the deformation of a body that is known prior to solving for the entire field.

Example 2.6 (Statically Indeterminate Bar).

PROBLEM STATEMENT:

Determine the reaction forces in on the bar shown in Figure 2.14.

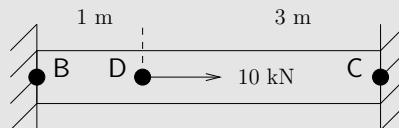


Figure 2.14: Example 2.6

KNOWNs:

- $L = 4$ m
- 10 kN point load, 1 m from left end
- Fixed supports on both ends

ASSUMPTIONS:

- Neglect end effects
- $D \ll L \implies$ Long and slender
- Linearly elastic material

ANALYSIS:

The member is subjected only to axial loads, so by including the assumptions, a *bar model* is appropriate for this problem. A free-body diagram of the bar is shown in Figure 2.15(a). Applying static equilibrium:

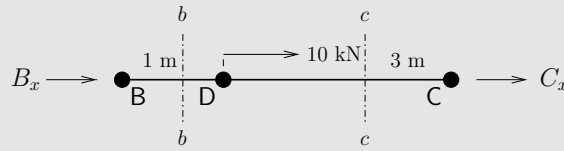
$$\sum F_x = 0 = B_x + C_x + 10 \quad (2.37)$$

The forces B_x and C_x are unknown, so equation (2.37) is not solvable. It is necessary to define a compatibility condition. In this case, both ends of the bar are fixed in place, so we can say that the displacement of point C relative to point B is zero.

$$\delta_{C/B} = 0 \quad (2.38)$$

The relative deflection is defined using equation (2.15):

$$\delta_{C/B} = \int_B^C \frac{N}{AE} dx = \int_B^D \frac{N_b}{AE} dx + \int_D^C \frac{N_c}{AE} dx \quad (2.39)$$



(a) Free-body diagram of entire bar

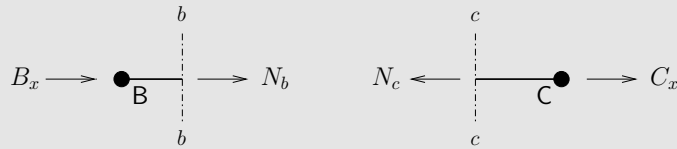
(b) Section $b - b$ (c) Section $c - c$

Figure 2.15: Free-body diagrams

Consider sections $b - b$ and $c - c$ (shown in Figures 2.15(b) and (c)) to determine the internal forces.

Section b – b:

$$\sum F_x = 0 = B_x + N_b \quad \Rightarrow \quad N_b = -B_x \quad (2.40)$$

Section c – c:

$$\sum F_x = 0 = C_x - N_c \quad \Rightarrow \quad N_c = C_x \quad (2.41)$$

The internal force in each segment is constant, so the integrals in equation (2.39) reduce to multiplication. Substituting the definitions of the internal forces into the compatibility condition:

$$\begin{aligned} \delta_{C/B} = 0 &= \frac{(-B_x)(L_b)}{AE} + \frac{(C_x)(L_c)}{AE} \\ \Rightarrow \quad B_x &= 3C_x \end{aligned} \quad (2.42)$$

We now have two equations in two unknowns, so can solve for the reactions. Solving equations (2.37) and (2.42) simultaneously:

$$\begin{aligned} B_x + C_x &= -10 \\ B_x - 3C_x &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} B_x &= -7.5 \text{ kN} \\ C_x &= -2.5 \text{ kN} \end{aligned} \quad (2.43)$$

Example 2.7 (Statically Indeterminate Bars).

PROBLEM STATEMENT:

Consider the bars shown in Figure 2.16. A circular aluminum rod is encased in a steel shell. The free ends of the rods are fused together and loaded with 200 kN. Determine the force the fixed support provides to each material.

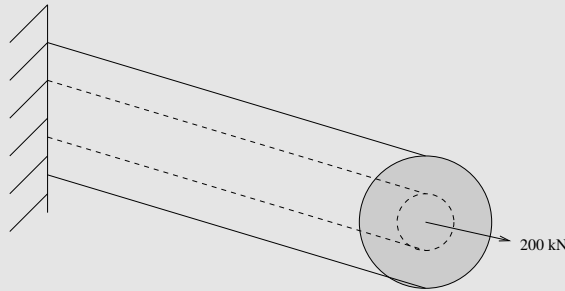


Figure 2.16: Example 2.7

KNOWNs:

- $D_a = 0.07 \text{ m} \rightarrow A_a = 0.00385 \text{ m}^2$

- $D_s = 0.08 \text{ m} \rightarrow A_s = 0.00118 \text{ m}^2$
- From Appendix B:
 - $E_a = 70 \times 10^9 \text{ N/m}^2$
 - $E_s = 200 \times 10^9 \text{ N/m}^2$
- Fixed support on left end for both materials
- Materials fused at right end
- Point load of 200 kN at right end

ASSUMPTIONS:

- Neglect end effects
- Members are long and slender

ANALYSIS:

At first glance, this problem appears to violate at least one of our most fundamental assumptions: this structure is not a homogeneous material. However, we can work through this by considering the structure as two separate members with some interaction/constraint between them. Considering each rod individually, the requirements for *bar analysis* are fully satisfied. Furthermore, the equilibrium equations must hold for any static body, regardless of constitution.

The free-body diagram is shown in Figure 2.17, where the subscript *a* refers to the aluminum rod and the subscript *s* refers to the steel. Applying the equation of static equilibrium:

$$\sum F_x = 0 = F_s + F_a + 200 \quad (2.44)$$



Figure 2.17: Free-body diagram

Again, equilibrium is insufficient to solve for the reactions and a compatibility condition is necessary. Because the two bars are fused at the free end, they

deform together. Therefore, we can define the compatibility condition:

$$\delta_a = \delta_s$$

$$\int_L \frac{N_a}{A_a E_a} dx = \int_L \frac{N_s}{A_s E_s} dx \quad (2.45)$$

This compatibility condition relates the deformation of the two bars to each other. It allows us to solve two bar problems simultaneously. In this case each material is subjected to a single point load, so sections are not necessary. The internal forces can be expressed in terms of the reaction forces.

$$N_a = -F_a \quad N_s = -F_s \quad (2.46)$$

Once again, the integrals reduce to multiplication. Substituting the definitions of the normal forces into equation (2.45) and simplifying:

$$\frac{-F_a L}{A_a E_a} = \frac{-F_s L}{A_s E_s} \quad (2.47)$$

The lengths of the two bars are equal and cancel. Substituting appropriate values of Young's modulus and area into equation (2.47), we find a second equation to use with equilibrium:

$$F_a = 1.14F_s \quad (2.48)$$

Combining with equilibrium the reaction forces are found to be:

$$F_a = -106.5 \text{ kN}$$

$$F_s = -93.5 \text{ kN} \quad (2.49)$$

2.4 Strain Energy

As discussed in Section 1.2.3, strain energy is the work related to deformation of a solid. Generally of greater interest is the strain energy density:

$$u = \int \sigma d\varepsilon \quad (2.50)$$

In the case of elastic deformation, this energy is fully-recoverable (see spring potential energy for similar behavior). For 1D linear elasticity, it is possible to develop quite simple analytical expressions for the strain energy density.

$$u = \frac{1}{2} \sigma \varepsilon + C \quad (2.51)$$

The constant of integration C must be zero because the strain energy density is zero when the material is unstrained. Substituting Hooke's law into equation (2.51) for either stress or strain yields:

$$u = \frac{E\varepsilon^2}{2} = \frac{\sigma^2}{2E} \quad (2.52)$$

Determination of the the total strain energy is as simple as integrating over the volume:

$$U = \int u dV \quad (2.53)$$

Due to the nature of bar problems, the strain energy density is constant throughout every cross-section. Integration over the area reduces to simple multiplication, leaving only integration over the length:

$$U = \int_L A(x)u dx \quad (2.54)$$

Example 2.8 (Strain Energy of a Bar).

PROBLEM STATEMENT:

Determine total strain energy and modulus of resilience of the bar from Example 2.5 if the yield stress is 200 MPa.

KNOWNs:

- All information from Example 2.5
- $\sigma_{xx} = 318,500(1 - x) \text{ N/m}^2$
- $S_y = 200 \times 10^6 \text{ N/m}^2$

ASSUMPTIONs:

- Neglect end effects

ANALYSIS:

Because this is a bar (as confirmed in the original problem), there is only axial stress present. The stress found in Example 2.5 varies along the length of the bar, so first compute the strain energy density at every point in the bar using equation (2.52) with the stress defined by equation (2.35).

$$u = \frac{\sigma^2}{2E} = \frac{318,500(1 - x)}{6 \times 10^8} = 0.00053(1 - x) \text{ J/m}^3 \quad (2.55)$$

Integrate u over the volume according to equation (2.54) to obtain the total strain energy:

$$U = \int_0^1 A(x)u \, dx = 0.00785 \int_0^1 0.00053(1-x) \, dx = 2.08 \times 10^{-6} \text{ J} \quad (2.56)$$

The modulus of resilience is defined in Section 1.2.3 as the total area under the elastic region of the stress strain curve. In other words, it is the strain energy density at yielding. Substituting the yield stress into equation (2.52):

$$u_r = \frac{S_y^2}{2E} = \frac{200 \times 10^6}{6 \times 10^8} = 0.3333 \text{ J/m}^3 \quad (2.57)$$

2.5 Thermal Effects

Recall the discussion of thermal strains in Section 1.2.3. In the case of bars, we are primarily interested in expansion in the axial direction. The strain is given by:

$$\varepsilon_{thermal} = \alpha \Delta T \quad (2.58)$$

where α is the linear coefficient of thermal expansion (a material property), and ΔT is the temperature change. According to equation (2.58), the absolute temperature of the material is not relevant, only its change from a given reference state. This is true for small changes in temperature. Remember, the coefficient of thermal expansion can be a function of temperature, which would result in a nonlinear relation for equation (2.58).

Although we are primarily interested in the axial strain, keep in mind that thermal expansion is a volumetric effect. Even for bars, there are still lateral thermal strains. For example, the change of diameter of a heated bar could be computed as:

$$\varepsilon_{thermal} = \alpha \Delta T = \frac{\Delta D}{D} \quad (2.59)$$

Equation (2.59) is true assuming, of course, that the bar is unconstrained and stress-free. If the bar is prevented from deforming, or heated irregularly, then a thermal stress field develops. This stress field is of particular interest in applications, but requires determination of the constraining forces. A bar must be constrained on both ends if expansion is to be prevented, so thermal stress problems become statically indeterminate problems. Just like purely mechanical problems, analysis of thermal stresses in bars will require compatibility conditions. The difference here is that there are two sources of deformation: thermal deformation due to the change of temperature, and mechanical deformation

caused by the reaction forces. Following determination of the compatibility condition, solving for thermal stresses in bars is straightforward.

Example 2.9 (Thermal Stresses).

PROBLEM STATEMENT:

Consider a bar fixed on both sides, as in Figure 2.13. The bar is heated from 300 K to 425 K, and made of aluminum. The diameter is 0.01 m. Determine the thermal stresses.

KNOWNs:

- From Appendix B
 - $\alpha = 23 \times 10^{-6} \text{ 1/K}$
 - $E = 70 \times 10^9 \text{ N/m}^2$
- $D = 0.01 \text{ m} \implies A = 7.854 \times 10^{-5} \text{ m}^2$
- $\Delta T = 125 \text{ K}$

ASSUMPTIONs:

- Neglect end effects
- Small strains
- Member is long and slender

ANALYSIS:

With the assumptions and load data, this problem satisfies the requirements for *bar analysis*. The bar is constrained on both ends; when thermal expansion occurs, forces from the supports prevent motion. Static equilibrium is insufficient to determine these two forces, so a compatibility condition is necessary. For this problem, the support forces are wholly due to thermal expansion (i.e., there are no other forces present), so the expansion due to temperature change and compression due to the mechanical forces must balance:

$$\delta_m = -\delta_t \quad (2.60)$$

where δ_m is mechanical deformation and δ_t is thermal deformation. Substituting definitions of these two terms:

$$\int_0^L \frac{N}{AE} dx = -\alpha \Delta T L \quad (2.61)$$

The only forces on the bar are the support forces R_x , so the internal normal force at any point in the bar is that constant value. Equation (2.61) therefore reduces to:

$$\frac{R_x L}{AE} = -\alpha \Delta T L \quad (2.62)$$

Substituting known values and solving:

$$R_x = -15.8 \text{ kN} \quad (2.63)$$

Finally, to determine the stress the internal normal force must be divided by the area.

$$\sigma = \frac{N}{A} = \frac{R_x}{A} = -201 \text{ MPa} \quad (2.64)$$

Thermal stresses in bars are limited to problems like Example 2.9. If the stresses are caused by the presence of a temperature gradient, the structure will not be a bar. Thermal expansion is volumetric; if the material is expanding unevenly, it will create a 3D stress field. Furthermore, if the bar is truly fixed at each end and unable to expand laterally, end effects may be significant and care should be taken to appropriately apply Saint-Venant's principle.

2.6 Superposition

Consider a function $T(\mathbf{x})$. This function is known as a **linear transformation** or **linear mapping** if it satisfies the following requirements:

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}) \\ T(c\mathbf{u}) &= cT(\mathbf{u}) \end{aligned} \quad (2.65)$$

where \mathbf{u} and \mathbf{v} are vectors in some space and c is a scalar. When the transformation T is linear, it may be decomposed into certain components and produce equivalent results. Some care is required, though. A *linear function* is not necessarily a *linear transformation*. This is well-illustrated using the second condition in equation (2.65) with a simple example:

$$f(x) = x + 3 \quad (2.66)$$

is obviously a linear function. However,

$$f(2x) = 2x + 3 \neq 2(x + 3) = 2f(x) \quad (2.67)$$

Therefore, it is not a linear transformation. The requirement that scalar multiples be distributable for linear transformations implies a **homogeneity** requirement (in the mathematical sense); the linear function cannot contain an isolated constant. Transformations such as (2.66) are called **affine**.

Linear transformations are useful in many mechanical applications because because many load-deflection or load-stress relations are linear. In other words, we can take advantage of the properties described by equation (2.65) in solving complex mechanics problems by considering individual sub-problems and summing the results. This process is known as **superposition**. There are some important points to consider before trying to use this principle:

1. Superposition can only be used when all quantities are linearly related. Linearity does not simply refer to the spatial description of quantities though. It often describes the relation between internal loads and deformation, for example. The internal normal force may be nonlinear in x , but it is still linearly related to the elongation of a bar, so superposition is applicable.
2. The material must not deform significantly from its initial configuration if superposition is to be used. There are several reasons for this. Most materials are only linearly elastic over a small range of strains. Strain itself is only linearly related to material displacement for small values. If either of these assumptions do not hold, then the linearity requirement is not satisfied. Even if strains are small though, a large enough change in geometry can affect how external loads are related to the internal field and deformation. Figure 2.18 shows an example of how this can be problematic. Even though the loads P_1 and P_2 only create small strains, the transverse force P_1 bends the beam down far enough that the axial load P_2 creates a moment. The distance between the wall and P_1 could even significantly change, which would change the moment caused by that force.

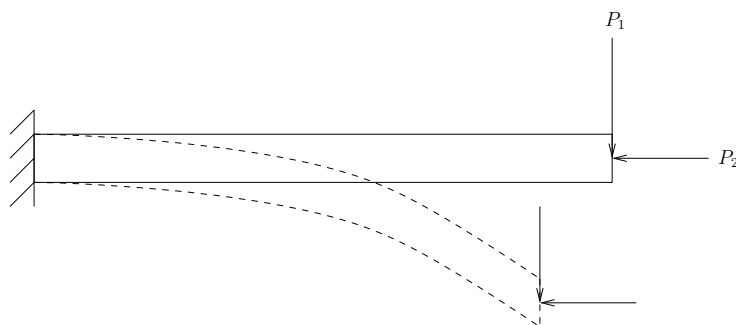


Figure 2.18: A structural member with significant deformation

3. Superposition is a general analytical tool that is not applicable only to bar problems. As long as deflections are small and load-deflection or load-stress relations are linear and homogeneous, then superposition may be used.

Example 2.10 (Verification of Superposition).**PROBLEM STATEMENT:**

Verify the principle of superposition by solving for deflection of the prismatic bar shown in Figure 2.19 under the action of both loads simultaneously, and comparing that solution with the sum of deflections of the bar under the actions of the applied loads individually.



Figure 2.19: Example 2.10

KNOWNs:

- Bar is 2 m long
- Fixed support at the left end
- 10 kN force applied at $x = 1$ m
- 5 kN force applied at $x = 2$ m
- Bar is prismatic (constant area)

ASSUMPTIONs:

- Neglect end effects
- Small strains
- Linearly elastic deformation
- Member is long and slender

ANALYSIS:

With the assumptions and load data, this problem satisfies the requirements for *bar analysis*. Analysis may proceed.

Begin with a free-body diagram of the entire bar, as shown in Figure 2.20(a). Apply static equilibrium to determine the reaction force R :

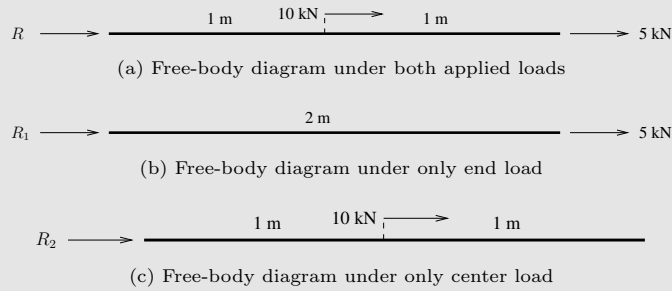


Figure 2.20: Problem free-body diagrams

$$\sum F_x = 0 = R + 10 + 5 \implies R = -15 \text{ kN} \quad (2.68)$$

To solve for deformation, determine the internal normal force distribution in this bar. Sections are necessary on either side of the 10 kN force:

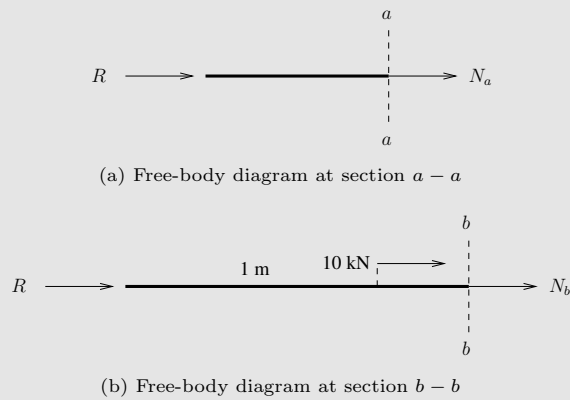


Figure 2.21: Section diagrams for Figure 2.20(a)

Apply static equilibrium at each section to find the internal normal force:

$$\begin{aligned} \text{Section } a - a: \quad \sum F_x = 0 = R + N_a &\implies N_a = 15 \text{ kN} \\ \text{Section } b - b: \quad \sum F_x = 0 = R + 10 + N_b &\implies N_b = 5 \text{ kN} \end{aligned} \quad (2.69)$$

Since the internal normal forces and area are constant, the bar equation reduces to equation (2.29) and total elongation of the bar is given by:

$$\delta = \frac{N_a L_a}{AE} + \frac{N_b L_b}{AE} = \frac{20 \text{ kN}\cdot\text{m}}{AE} \quad (2.70)$$

We repeat this process for the case subjected only to the load at end of the bar (Figure 2.20(b)). From static equilibrium:

$$R_1 = -5 \text{ kN} \quad (2.71)$$

The internal normal force in this case is the same as the applied point load, and the deformation of the bar is found to be:

$$\delta_1 = \frac{PL}{AE} = \frac{(5 \text{ kN})(2 \text{ m})}{AE} = \frac{10 \text{ kN-m}}{AE} \quad (2.72)$$

Finally, determine the forces and deformation of the bar subjected only to the load at $x = 1 \text{ m}$ (Figure 2.20(c)). Begin with equilibrium to find the external reaction R_2 :

$$\sum F_x = 0 = R_2 + 10 \implies R_2 = -10 \text{ kN} \quad (2.73)$$

Since the load state on the bar changes along its length, the internal normal force does as well. Taking sections on either side of the 10 kN load, as shown in the free-body diagrams in Figure 2.22, and applying equilibrium:

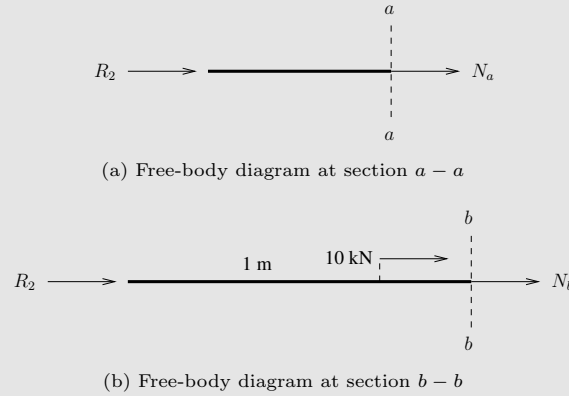


Figure 2.22: Section diagrams for Figure 2.20(c)

$$\text{Section } a - a : \sum F_x = 0 = R_2 + N_a \implies N_a = 10 \text{ kN} \quad (2.74)$$

$$\text{Section } b - b : \sum F_x = 0 = R_2 + 10 + N_b \implies N_b = 0 \text{ kN}$$

Once again, apply equation (2.29) to determine the deformation.

$$\delta_2 = \frac{N_a L_a}{AE} + \frac{N_b L_b}{AE} = \frac{10 \text{ kN-m}}{AE} \quad (2.75)$$

Superposition the individual deformation terms:

$$\delta_1 + \delta_2 = \frac{10 \text{ kN-m}}{AE} + \frac{10 \text{ kN-m}}{AE} = \frac{20 \text{ kN-m}}{AE} = \delta \quad (2.76)$$

The total deformation of the bar is identical to the sum of the deformations caused by the individual loads, confirming that superposition is valid for this problem. If the deformation is a linear transformation of the applied load, regardless of the nature of the load, then superposition is applicable. Note that this applies not only to the elongation of the entire bar, but also to the deformation, strain, and stress at any point in the bar.

Besides reconstructing complex problems in simpler forms, superposition has several more specific uses in bar analysis. The first is an alternative method for solving certain statically indeterminate problems. By removing a constraint, the problem becomes statically determinate and may be solved for deflection. If the force from the constraint is applied to the bar, it must deflect the bar back to the constrained position. If these two results are superpositioned, it solves the original problem. Logically, this is equivalent to formulating the compatibility condition, but may be algebraically simpler in some cases.

Example 2.11 (Superposition and Static Indeterminacy).

PROBLEM STATEMENT:

Solve the problem of Example 2.6 using superposition rather than a compatibility condition.

KNOWNs:

- $L = 4$ m
- 10 kN point load, 1 m from left end
- Fixed supports on both ends

ASSUMPTIONs:

- Neglect end effects
- $D \ll L$
- Linearly elastic material

ANALYSIS:

As previously discussed, this problem reduces to a *bar analysis*. Begin this problem with the free-body diagram.

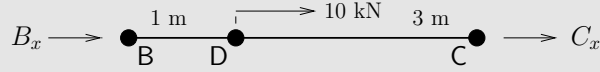


Figure 2.23: Complete free-body diagram

This problem is clearly statically indeterminate. Instead of defining a compatibility condition, however, we will reduce the problem to a statically determinate case. We do this by “releasing” the right end of the bar; the force C_x is momentarily neglected.

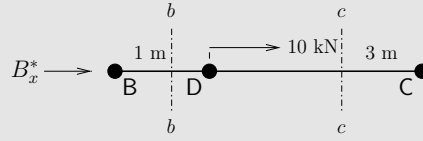


Figure 2.24: Free-body diagram with release

From the free-body diagram in Figure 2.24, we can directly solve for the deflection at point C. Begin by solving for the reaction (note that $B_x^* \neq B_x$).

$$\sum F_x = 0 = B_x^* + 10 \implies B_x^* = -10 \text{ kN} \quad (2.77)$$

It is clear by direct observation that the internal normal forces at sections $b - b$ and $c - c$ are:

$$\begin{aligned} N_b^* &= 10 \text{ kN} \\ N_c^* &= 0 \text{ kN} \end{aligned} \quad (2.78)$$

The deflection of the bar at point C is equivalent to the total elongation of the bar. From equation (2.29):

$$\delta^* = \frac{(N_b^*)(L_b)}{AE} + \frac{(N_c^*)(L_c)}{AE} = \frac{10}{AE} \quad (2.79)$$

Now consider the bar subjected only to the force C_x , as shown in Figure 2.25.

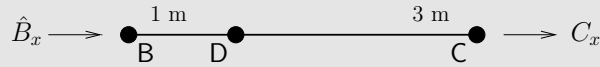


Figure 2.25: Free-body diagram with reaction

Once again, the internal normal force is clear from direct observation.

$$\hat{N} = C_x \quad (2.80)$$

The primary difficulty here is that C_x is unknown. However, the deformation can still be determined in terms of that quantity.

$$\hat{\delta} = \frac{\hat{N}L}{AE} = \frac{4C_x}{AE} \quad (2.81)$$

At this point, deformation due to each load has been fully defined. Since deformation is linear in the internal normal forces, the total deformation of the bar is the superposition of the individual deformation fields.

$$\delta = \delta^* + \hat{\delta} = \frac{1}{AE}(10 + 4C_x) \quad (2.82)$$

From the constraints shown in Figure 2.16, it is clear that the total elongation of the bar must be $\delta = 0$. Using this with the superpositioned results:

$$\delta = 0 = \frac{1}{AE}(10 + 4C_x) \implies C_x = -2.5 \text{ kN} \quad (2.83)$$

From static equilibrium, the reaction at the left end is:

$$\sum F_x = 0 = B_x + C_x \implies B_x = -7.5 \text{ kN} \quad (2.84)$$

These reaction forces exactly match the results of Example 2.6 determined using a compatibility condition.

DISCUSSION:

- Use of superposition for statically indeterminate problems requires separate consideration of knowns and unknowns.
- Even with superposition, a compatibility condition was required ($\delta = 0$), although it is perhaps simpler than direct application of compatibility conditions.
- Superposition only holds for linear transformations. The compatibility conditions are expressions of physical restrictions and do not depend on linearity. Superposition is therefore somewhat more limited in application.
- In this problem, δ^* and $\hat{\delta}$ are used to denote the behavior of the two sub-problems. This is not a specific notation, only something to mark different terms.

The principle of superposition also shows clear usefulness in thermal expansion problems. Temperature is linearly related to elongation (for small temperature changes), as is force, therefore superposition is a valid option in these

problems. If a bar is constrained and cannot expand, there must be a resultant force preventing that motion. Thus, in the statically indeterminate problem of Example 2.11, the deformation due to thermal expansion can be summed with the compression due to the constraints.

2.7 Differential Relation for Bars*

From Section 1.4, we know that deformation of a bar can be described by a differential equation. Recall equation (1.113):

$$F^b + \frac{d}{dx} \left(AE \frac{du_x}{dx} \right) = 0 \quad (2.85)$$

This equation holds for any one-dimensional problem. Consider its application to the bar shown in Figure 2.26.

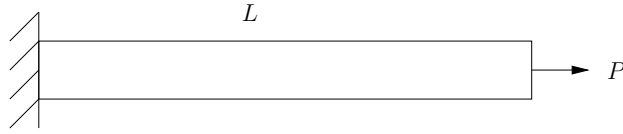


Figure 2.26: Bar fixed at left end, point load at the right

In this case, we can make the following observations:

- Assuming L is much greater than the lateral dimensions, the bar is long, slender, straight, and loaded axially, satisfying the conditions for bar theory.
- The cross-sectional area is constant, i.e., $A \neq A(x)$.
- The left boundary is fixed in place.
- A point load P is applied to the right end of the bar.
- There is no distributed load, so $F^b = 0$.

With these conditions, it is possible to simplify (2.85).

$$AE \frac{\partial^2 u_x}{\partial x^2} = 0 \quad (2.86)$$

This differential equation is solvable through direct integration. Integrating once:

$$AE \frac{\partial u_x}{\partial x} = C_1 \quad (2.87)$$

And again:

$$AE u_x = C_1 x + C_2 \quad (2.88)$$

Determination of the constants of integration require boundary conditions. From Figure 2.26, the bar is fixed at the left end, providing a simple boundary condition.

$$u_x(0) = 0 \quad (2.89)$$

A force P is applied to the right end of the bar. As force does not appear in the differential equation (2.86) or its solution (2.88), some additional work is required to make use of this information. Recall that force is related to stress, which in turn is related to strain, and finally to displacement.

$$\frac{P}{A} = \sigma(L) = E\varepsilon(L) = E \left. \frac{\partial u_x}{\partial x} \right|_L \implies u'_x(L) = \frac{P}{AE} \quad (2.90)$$

The left BC may be substituted directly into (2.88).

$$AEu_x(0) = 0 = C_1(0) + C_2 \implies C_2 = 0 \quad (2.91)$$

The right BC must be substituted into the derivative of (2.88).

$$\begin{aligned} u'_x &= \frac{C_1}{AE} \\ u'_x(L) &= \frac{P}{AE} \implies C_1 = P \end{aligned} \quad (2.92)$$

With the constants of integration defined, the solution of the differential equation reduces to:

$$u_x(x) = \frac{P}{AE}x \quad (2.93)$$

The displacement of any point in the bar is given by (2.93). Since the left edge is fixed, the displacement of a point is equivalent to the elongation of the bar between that point and the left boundary. In other words, (2.93) may be written as:

$$\delta(x) = \frac{P}{AE}x \quad (2.94)$$

which at the right end of the bar becomes:

$$\delta = \frac{PL}{AE} \quad (2.95)$$

This is equivalent to the result found in this chapter utilizing an independent derivation of bar theory. Arriving at the same solution through different means should give us confidence in the consistency of our work, and in fact, it should be expected in this case; the two representations of the bar model (1.113) and (2.6) are both the result of application of Newton's second law. They are simply different representations of the same physics.

2.8 Chapter Summary

Bars are perhaps the simplest possible surrogate model in solid mechanics. However, this simplistic model is only applicable when the following assumptions and restrictions hold:

1. The material is continuous, homogeneous, isotropic, and linearly elastic
2. The structure is long, slender, and straight
3. The member undergoes only small deformation
4. The structure is loaded only by forces oriented along its longitudinal axis
5. The forces are approximately centered on the axis of the member
6. Area changes occur gradually

In such a case, the mechanical analysis of the structure reduces to a purely one-dimensional problem. The bar elongates or compresses, as expressed by the average normal strain. Following force equilibrium and Saint-Venant's principle, the average normal stress is equivalent to the actual stress at any point in the cross-section. By applying these simplifications to Hooke's law, the internal normal force is directly related to the change in length of the bar.

Change in length may also be determined for specific segments of the bar instead of the entire object. A useful notation for describing such deformation $\delta_{B/A}$ explicitly defines the motion of point B relative to point A. This concept of relative motion is particularly useful in determining the compatibility conditions necessary for analysis of statically indeterminate bars. Compatibility conditions are also useful in wiggling around the homogeneity requirement for certain problems; for a well-defined inter-material boundary, a compatibility condition can be used to decompose a single bar made of multiple materials into several connected but individual bar problems.

Basic concepts of thermal and energy effects are also explored quite readily using bar analysis. A uniformly heated bar expands uniformly in the absence of an external force field. If the bar is constrained from moving, the constraints exert forces on the bar and a thermal stress field develops. The stress field in bars can be directly analyzed using compatibility conditions or superposition. The expansion is volumetric, so temperature change will change lateral dimensions in addition to the length. The strain energy and strain energy density are measures of work stored in the material. Stress is the measure of force, and strain is a measure of motion; used together they can define a sort of potential energy of the loads on the material.

The principle of superposition is a mathematical tool that shows exceptional usefulness in mechanical analysis. Superposition is an extension of the concept of a linear transformation. If functions relating deformation and some disturbances (force, temperature, etc.) are linear and homogeneous, then deformation from each effect can be computed individually, then summed to give the total. The

total deformation (or stress, or other quantity of interest) is exactly equivalent to that found considering all disturbances simultaneously. Superposition may only be applied when the deformation of the material is small. Aside from large deformation introducing nonlinearities, it is possible that the loads themselves could behave differently depending on the deformation due to other loads; in such cases the deformation fields due to each individual disturbance are coupled and cannot be separated.

2.8.1 List of Symbols

σ_{xx} :	Normal axial stress
ε_{xx} :	Normal axial strain
N :	Internal normal force
A :	Cross-sectional area
δ :	Elongation of the bar
$\delta_{b/a}$:	Motion of point b relative to point a
E :	Young's modulus
k :	Extensional stiffness
L :	Undeformed length of bar segment
u :	Strain energy density
U :	Total strain energy
ΔT :	Change in temperature
α :	Coefficient of thermal expansion
D :	Cross-sectional dimension of bar

Chapter 3

Shaft Analysis

With a basic understanding of how surrogate models are developed and used from Chapter 2, we proceed to a slightly more complicated model. In this chapter we investigate **shafts**, which are 1D structural members subjected to axial torques. Due to this important difference, it is possible to investigate some very interesting phenomena using shafts that were absent in bars, but with similar levels of simplicity.

3.1 Fundamentals of Shaft Models

Similar to bars, shafts are 1D models. Many of the basic assumptions for shaft are therefore the same, with the exception of the loading:

1. The structure is loaded only by torques oriented along its longitudinal axis
2. The torques are approximately centered on the axis of the member
3. The structure is long, slender, and straight
4. Area changes occur gradually

Initial discussion of these assumptions is identical to that in Section 2.1. Some additional discussion is required for the torque at this point. The torque (as moments about the longitudinal axis are often called) is a pure moment. Physically, these moments are created by **couple forces**, as shown in Figure 3.1. It is critical that the forces form a couple. A single force on either end of the wrench in Figure 3.1 would still create a moment, but that force would be unbalanced and would result in a transverse load on the shaft.

While distributed torques exist, we generally do not treat them in the same sense as distributed forces. In bars, we have a distributed load over the cross-section that reduces analytically to a point load. The couple represents the physical load in the case of the shafts, while the pure torque is the model. If the

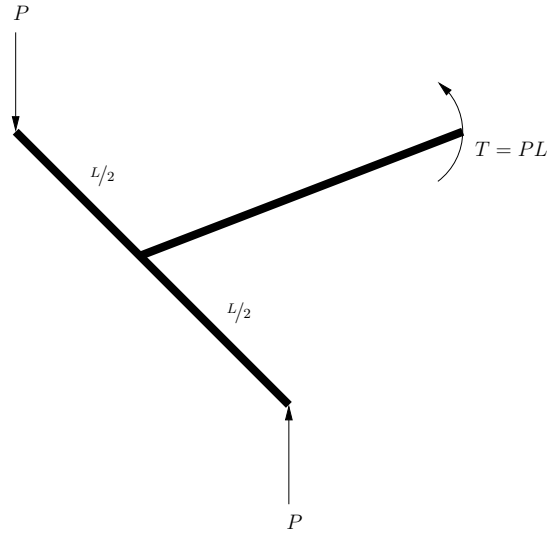


Figure 3.1: Couple forces creating a torque

couple forces are distributed along the length of the shaft, then the torque due to that couple would also be distributed. However, it is much more common for torques to be applied by relatively isolated forces (e.g., teeth on a gear) that are adequately represented by point loads. Near the point of application of the torque, more exotic loading and deformation is expected. For example, the handles of the wrench in Figure 3.1 may bend, providing a locally complex stress and deformation field. However, following Saint-Venant's principle, these disturbances fade quickly with distance from the point of application.

With these assumptions, shafts also reduce to one-dimensional problems along the length of the member. The free-body diagram is typically represented as a single line, and cross-sectional dimensions are analytical parameters. Although the problem looks identical to a bar (with double-headed vectors instead), it is not a case of uniaxial deformation. A moment about the longitudinal axis of a shaft does not cause motion in the direction of that axis; rather, it results in *twisting* in the cross-sectional plane.

3.2 Torsion of Circular Shafts

The mode of deformation exhibited by shafts is called **torsion**. As mentioned previously, this is a twisting motion about the longitudinal axis of the shaft. The magnitude of twisting is measured as a change of angle from the initial configuration. The shaft shown in Figure 3.2(a) is general, with some torque applied along its length and cross-sectional geometry defined as a function of x . The **angle of twist** ϕ is measured in the cross-section (parallel to the yz -plane), positive counterclockwise. In this case, $\phi = 0$ at the fixed boundary and

reaches some maximum value at the free end, so clearly $\phi = \phi(x)$. In order for this model to reduce to a 1D case, the angle of twist cannot be a function of y and z (or r and θ). As shown in Figure 3.2(b), this is most easily achieved by requiring the shaft to be *circular* with radius R . If this condition is met, then ϕ is uniform across each cross-section.

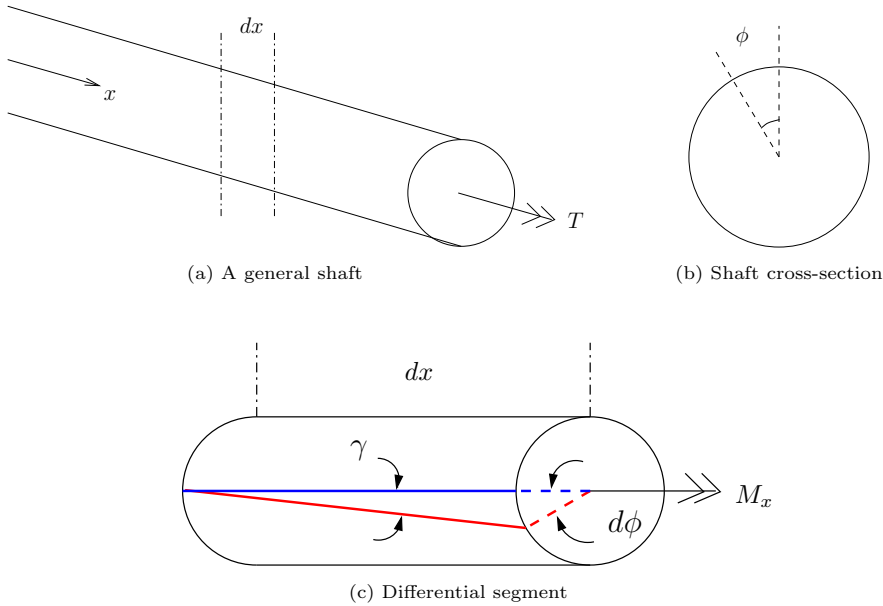


Figure 3.2: Shaft schematic

Much like in bar analysis, to develop a general shaft model we must consider a small segment of the shaft. The twist of this differential segment is $d\phi$, shown with dashed lines on the face of the section in in Figure 3.2(c). A line drawn along the length of the shaft would be horizontal prior to deformation (blue solid line), but after the shaft twists the line would have a different orientation (red solid line). The horizontal line and the cross-section are initially perpendicular. If the cross-section does not warp, then the angle between the horizontal and the deformed line measures the change of angle between the initially perpendicular lines. That is, the angle shown on the side of the shaft in Figure 3.2(c) is the *shear strain*!

Note the condition for the angle shown in Figure 3.2(c) to be the shear strain: that the cross-section does not warp. We are assuming that it only twists, but under what conditions does that hold? If we have restricted ourselves to circular shafts, are there circumstances in which the angle of twist becomes coupled to other deformation? In short, if the shaft deforms significantly (we will define exactly what that means shortly), then the length of the shaft could change, as could the radius (imagine a soda can that you have twisted sharply to crush). We assume that the shaft undergoes only small deformation then to isolate

the twisting motion (an assumption already present in all of our analyses as discussed in Section 1.3).

The arclength swept out by the angle of twist is given by

$$s = Rd\phi \quad (3.1)$$

With the assumption of small deformation, this circular arc is approximately a straight line segment. Simple trigonometry can then relate the arclength to the shear strain.

$$\sin(\gamma) = \frac{s}{dx} = \frac{Rd\phi}{dx} \quad (3.2)$$

Small deformation further simplifies the model from here. The sine of a small angle is approximately the angle itself, so equation (3.2) reduces to:

$$\gamma = R \frac{d\phi}{dx} \quad (3.3)$$

Equation (3.3) relates the shear strain to the **rate of twist** $\frac{d\phi}{dx}$. Note that the simplification from equation (3.2) to (3.3) defines small deformation for our shaft problem. If the rate of twist is small enough that the approximation error in $\sin(\gamma) \approx \gamma$ is negligible, then the assumption is satisfied. For example, $\sin(0.1) = 0.0998$, an approximation error of 0.2%.

Strictly speaking, the shear strain defined by (3.3) is only the strain on the surface of the shaft. As the radial distance reduces, the arclength decreases and in turn the shear strain decreases, eventually reaching zero at the centerline of the shaft. Let $r \in [0, R]$ be the radial variable, then the shear strain is more generally defined as:

$$\gamma = r \frac{d\phi}{dx} \quad (3.4)$$

The peak value is at the outer radius of the shaft though, so (3.3) is most often used. The shear stress is related to the shear strain by Hooke's law for shear:

$$\tau = G\gamma = Gr \frac{d\phi}{dx} \quad (3.5)$$

The maximum value of shear stress due to torsion is also located on the outer surface of the shaft, and is given by:

$$\tau_m = GR \frac{d\phi}{dx} \quad (3.6)$$

Solving (3.6) for the rate of twist and substituting into (3.5), we can solve for the shear stress as a fraction of its own maximum value:

$$\tau = \frac{r}{R} \tau_m \quad (3.7)$$

The shear stress τ is a function of the radial variable r , but the maximum value τ_m is a constant. This shear stress acts parallel to the radial surface of the

shaft, parallel and perpendicular to the cross-section, as illustrated in Figure 3.3(a). Because the shear stress varies in the radial direction, it can be shown that components of the stress also act on the cross-section (Figure 3.3(b)). With this, shear stress and strain are fully-defined in terms of angle of twist, but the angle of twist is still unknown. Intuitively, ϕ (and therefore all of the other quantities) must depend upon the applied load.

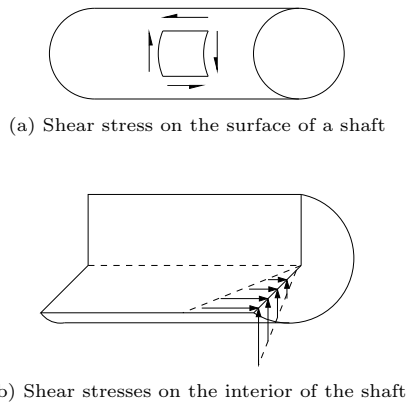


Figure 3.3: Shaft stresses

Now consider the cross-section of the shaft. The moment is defined as force times its perpendicular distance. We have a measure of force, the shear stress. This stress is perpendicular to the radius at every point. However, the stress varies in the radial direction, so it is not possible to simply consider a single value. Instead, consider the stress acting on a small area element on the cross-section, as shown in Figure 3.4:

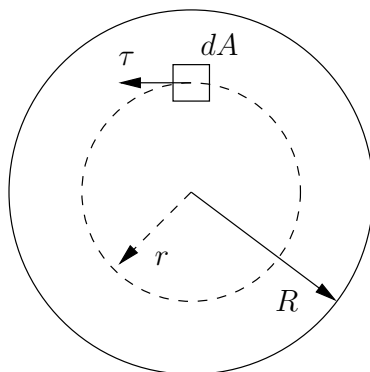


Figure 3.4: Moments on the shaft cross-section

$$dM = (r)(\tau dA) \quad (3.8)$$

To find the total moment about the axis of the shaft, sum all of the differential moments. The total moment is the internal torque acting on that cross-section.

$$T = \int dM = \int r\tau dA \quad (3.9)$$

r , τ , and dA are all related, so the integration in equation (3.9) is not trivial. It is possible to substantially simplify the integral by substituting (3.7) into (3.9):

$$T = \int r \left(\frac{r}{R} \tau_m \right) dA = \frac{\tau_m}{R} \int r^2 dA \quad (3.10)$$

The maximum shear stress and outer radius are constants, leaving only the integral of r^2 over the area. As described in Section 1.1.5.2, this quantity is the **polar second area moment**. It is a purely geometric quantity, a property of the size and shape of the shaft. Typically denoted J , the polar second area moment is determined for a circle by converting the integral over the area to an integral over the radius:

$$\begin{aligned} J &= \int r^2 dA = \int_0^R r^2 (2\pi r dr) \\ &= \frac{\pi r^4}{2} \Big|_0^R \\ &= \frac{\pi R^4}{2} \end{aligned} \quad (3.11)$$

As shafts are often measured in terms of diameter rather than radius, it may be convenient to express J in those terms.

$$J = \frac{\pi D^4}{32} \quad (3.12)$$

Conveniently for us, we currently only consider circular shafts, so the expression derived for J in equations (3.11) and (3.12) is broadly applicable and integration over another shape to determine J is rarely necessary.

Replacing the integral in (3.10) with J and using (3.7) to convert back to the variable stress, we are able to solve for shear stress τ in terms of the internal torque T :

$$\tau = \frac{Tr}{J} \quad (3.13)$$

This is obviously maximum when $r = R$. Equation (3.13) is often called the **torsion formula**. The shear stress varies linearly in the radial direction. With shear stress defined directly in terms of torque, we can substitute equation (3.13) into (3.5) to solve for angle of twist:

$$Gr \frac{d\phi}{dx} = \frac{Tr}{J} \quad (3.14)$$

$$\frac{d\phi}{dx} = \frac{T}{JG} \quad (3.15)$$

Solving this equation:

$$\phi = \int_L \frac{T}{JG} dx \quad (3.16)$$

This is the deformation equation for shafts, relating torque directly to angle of twist. It is completely analogous (and identical in form) to the bar equation (2.6). Deformation (ϕ and δ) is related to load (T and N), geometry (J and A), and material (G and E). Also similarly to bars, we can express angle of twist in terms of relative motions:

$$\phi_{b/a} = \int_a^b \frac{T}{JG} dx \quad (3.17)$$

3.2.1 Internal torques

Determination of internal torques for shafts is procedurally identical to finding internal normal forces for bars. A simple example is shown in the following. All the details and discussion follow precisely from Section 2.2.

Example 3.1 (Internal Torques).

PROBLEM STATEMENT:

Determine the internal torques in the shaft shown in Figure 3.5(a).

KNOWNs:

- Point loads are applied at locations Q, R, and S

ANALYSIS:

To determine the internal torques, sections must be taken every time the external loading changes. Therefore, consider section $a - a$ between Q and R and section $b - b$ between R and S.

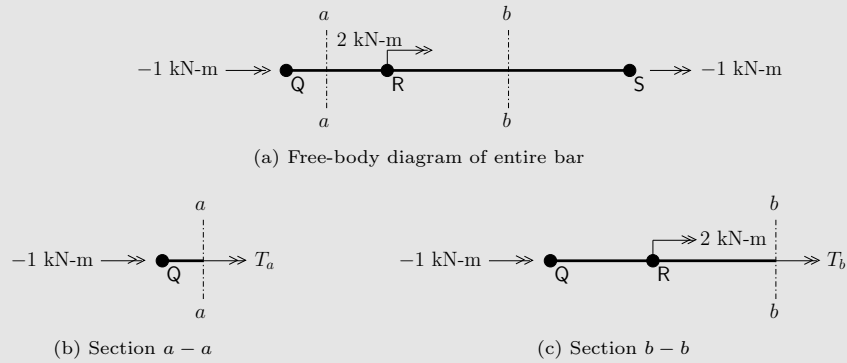


Figure 3.5: Example 3.1

Apply equilibrium at each section to determine the value of the internal torque at that location.

Section $a - a$:

$$\sum M_{O_x} = 0 = -1 + T_a \quad \implies \quad T_a = 1 \text{ kN-m} \quad (3.18)$$

Section $b - b$:

$$\sum M_{O_x} = 0 = -1 + 2 + T_b \quad \implies \quad T_b = -1 \text{ kN-m} \quad (3.19)$$

Because the external loading only changes at points Q, R, and S, the internal torque is constant between them. Therefore, T_a describes the internal torque at every location between Q and R, and T_b is the internal torque at every point between R and S. Thus, the internal torque can be expressed as a piecewise function:

$$T(x) = \begin{cases} 1 \text{ kN-m} & Q < x < R \\ -1 \text{ kN-m} & R < x < S \end{cases} \quad (3.20)$$

Note the sign change in T_a and T_b . Positive internal torques are counterclockwise, whereas negative internal torques are clockwise (as per the right hand rule). The torque field in this bar transitions from counterclockwise at point R. The left part of the shaft is being twisted in the opposite direction of the right part of the shaft.

As mentioned previously, this problem is completely analogous to Example 2.1 in Chapter 2. Mathematically, bar and shaft analysis are identical. The key difference between the two is the physical nature of the deformation field.

3.2.2 Application of the torsion formula

Solution of torsion problems is generally quite direct, but care must be taken to ensure that the assumptions used to derive the shaft model are satisfied. Particularly, keep in mind that equations (3.13) and (3.16) were derived assuming a *circular* cross-section and small deformation. The total angle of twist may be relatively large (depending on the length), but the rate of twist must be small enough that the sine approximation is valid. It is an easily checked condition for most problems, and the most common culprit for inadequate shaft analysis.

Example 3.2 (Drive Shaft).

PROBLEM STATEMENT:

Consider a driveshaft for a medium sized automobile, as shown in Figure 3.6. The shaft is 1.5 m long, and is tubular with an outer diameter of 60 mm. The shaft is aluminum. Both ends are subjected to a 475 N-m torque. Determine (a) the maximum stress and angle of twist if the inner diameter is 40 mm and (b) the maximum value of the inner diameter if the shear stress on the surface of the shaft is 170 MPa.

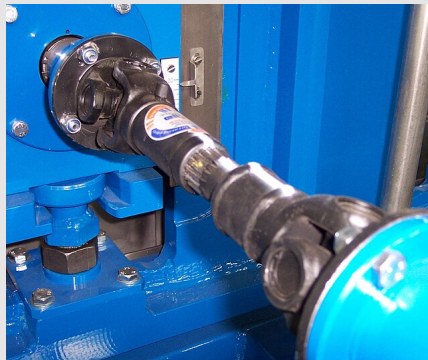


Figure 3.6: Example 3.2
(IP83, [Wikimedia Commons](#), [Creative Commons Attribution 3.0](#))

KNOWNs:

- $R_o = 0.03$ m
- $L = 1.5$ m
- $M = 475$ N-m
- From Appendix B, $G = 26$ GPa
- For part (a), $R_i = 0.02$ m

- For part (b), $\tau = 170$ MPa

ASSUMPTIONS:

- Neglect end effects
- Small deformation (verified after analysis)

ANALYSIS:

This structure is long and slender, loaded only by torques, and has a circular cross-section. With the assumptions, this satisfies all requirements for *shaft analysis*. Begin with a free-body diagram of the shaft.

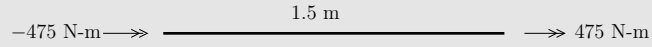


Figure 3.7: Free-body diagram

There are no changes in loading along the length of the shaft, so the internal torque is constant throughout the member.

$$T = M = 475 \text{ N-m} \quad (3.21)$$

Part (a):

To find the stress and angle of twist, employ the torsion and twist formulas (3.13) and (3.16). To use either, the polar second area moment is necessary.

$$J = \frac{\pi(R_o^4 - R_i^4)}{2} = \frac{\pi(0.03^4 - 0.02^4)}{2} = 1.02 \times 10^{-6} \text{ m}^4 \quad (3.22)$$

This value appears very low, but that is a relic of the units. The diameter of the shaft is a fraction of a meter, so when raised to the fourth power, becomes much smaller. Substituting values into (3.13):

$$\tau = \frac{TR}{J} = \frac{(475)(0.03)}{1.02 \times 10^{-6}} = 139.6 \text{ MPa} \quad (3.23)$$

This is the peak stress in the shaft, at the outer surface of the material. The angle of twist is found using equation (3.16); since the internal torque and geometry are constant, the integration reduces to multiplication with the length.

$$\phi = \int_L \frac{T}{JG} dx = \frac{TL}{JG} = \frac{(475)(1.5)}{(1.02 \times 10^{-6})(26 \times 10^9)} = 0.027 \text{ radians} \quad (3.24)$$

The final step in the analysis is to confirm that the shaft undergoes small deformation. This is evaluated using the rate of twist, defined in (3.15).

$$\frac{d\phi}{dx} = \frac{T}{JG} = 0.018 \text{ rad/m} \quad (3.25)$$

This rate of twist satisfies the small angle $\sin(\theta) \approx \theta$ approximation. Therefore, the assumption of small deformation was valid.

Part (b):

For this problem, the shear stress on the outer surface of the shaft is given. Rearranging equation (3.13), we solve for the polar second area moment.

$$J = \frac{TR}{\tau} = \frac{(475)(0.03)}{170 \times 10^6} = 8.382 \times 10^{-8} \text{ m}^4 \quad (3.26)$$

With this information, it is possible to solve equation (3.11) for the internal radius.

$$R_i = \left(R_o^4 - \frac{2J}{\pi} \right)^{1/4} = 0.0295 \text{ m} \quad (3.27)$$

DISCUSSION:

- The material in part (b) is designed to be quite thin (half a millimeter). Pure torsion (in the absence of other loads or deformation) induces relatively low stresses in the material.
- It may be necessary to determine the stress or deformation of a shaft. On the other hand, in many applications there is a known limit on the stress, and structural members are designed to approach that limit. Both parts of the problem may see practical usage.
- On a practical note, driveshafts are often designed to fail at lower torques than the transmission or the engine. They are substantially cheaper and easier to replace; early failure of the driveshaft prevents damage to more costly parts. In this example, the value stress given is the shear yield stress of aluminum, so this shaft is designed to fail at a torque of 475 N-m.

3.3 Statically Indeterminate Shafts

As previously mentioned in Sections 1.1.3 and 2.3, static indeterminacy results from overconstraining a structure. In the case of shafts, the reactions take

the form of torques, and the sum of moments is insufficient to determine the unknowns. The procedure for overcoming this obstacle is identical to that for bars: we must determine a compatibility condition from our knowledge of the deformation field prior to solution.

In fact, as the twist formula is completely analogous in form and function to the bar equation, the entire process follows directly. In many cases, if we simply replace forces with torques, we even get numerically identical results. Some additional care is required in the case of shafts, however. There is one additional requirement that we have imposed on our shaft problems that was not present in our study of bars: the cross-section of the shaft must be *circular*. While this perhaps seems trivial at first, it is important to keep in mind, as shown in the following example.

Example 3.3 (A Non-shaft).

PROBLEM STATEMENT:

Solve for the unknown reaction torques in cylindrical members A and B shown in Figure 3.8. The shafts are identical in dimension and material. They are anchored to a fixed support at the left boundary, and fused to a plate at the right boundary. A torque is applied to the plate.

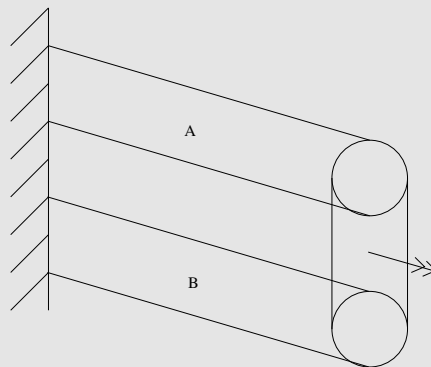


Figure 3.8: Example 3.3

ANALYSIS:

The first step is to draw a free-body diagram and identify the unknowns. As shown in the diagram below, the applied torque on the end plate must balance with the reactions at points A and B. These reactions are not necessarily torques, however. If the endcap rotates, it will induce a bending motion in members A and B and produce tangential forces. Even though members A and B have circular cross-sections, are long and slender, and the system is subjected to a torque about its centerline, the individual members are not shaft problems, and cannot be solved as such.

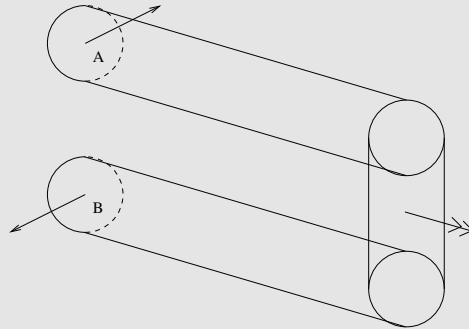


Figure 3.9: System free-body diagram

Example 3.4 (Statically Indeterminate Shaft).**PROBLEM STATEMENT:**

Determine the reaction forces in on the shaft shown in Figure 3.10.

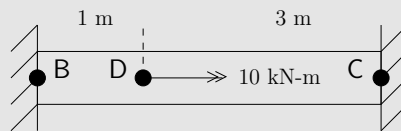


Figure 3.10: Example 3.4

KNOWNs:

- $L = 4$ m
- 10 kN-m point load, 1 m from left end
- Fixed supports on both ends

ASSUMPTIONs:

- Neglect end effects
- $D \ll L \implies$ Long and slender
- Linearly elastic material

ANALYSIS:

The member is subjected only to torques, so by including the assumptions, a *shaft model* is appropriate for this problem. A free-body diagram of the bar is shown in Figure 3.11(a). Applying static equilibrium:

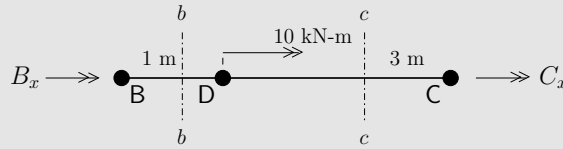
$$\sum M_{O_x} = 0 = B_x + C_x + 10 \quad (3.28)$$

The torques B_x and C_x are unknown, so equation (3.28) is not solvable. It is necessary to define a compatibility condition. In this case, both ends of the shaft are fixed in place, so we can say that the twist of point C relative to point B is zero.

$$\phi_{C/B} = 0 \quad (3.29)$$

The relative twist is defined using equation (3.17):

$$\phi_{C/B} = \int_B^C \frac{T}{JG} dx = \int_B^D \frac{T_b}{JG} dx + \int_D^C \frac{T_c}{JG} dx \quad (3.30)$$



(a) Free-body diagram of entire bar

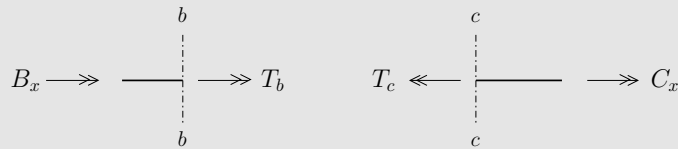
(b) Section $b - b$ (c) Section $c - c$

Figure 3.11: Free-body diagrams

Consider sections $b - b$ and $c - c$ (shown in Figures 3.11(b) and (c)) to determine the internal torques.

Section $b - b$:

$$\sum M_{O_x} = 0 = B_x + T_b \quad \implies \quad T_b = -B_x \quad (3.31)$$

Section $c - c$:

$$\sum M_{O_x} = 0 = C_x - T_c \quad \implies \quad T_c = C_x \quad (3.32)$$

The internal torque in each segment is constant, so the integrals in equation (3.30) reduce to multiplication. Substituting the definitions of the internal torques into the compatibility condition:

$$\begin{aligned}\phi_{C/B} = 0 &= \frac{(-B_x)(L_b)}{JG} + \frac{(C_x)(L_c)}{JG} \\ \implies B_x &= 3C_x\end{aligned}\quad (3.33)$$

We now have two equations in two unknowns, so can solve for the reactions. Solving equations (3.28) and (3.33) simultaneously:

$$\begin{aligned}B_x + C_x &= -10 & \implies & B_x = -7.5 \text{ kN-m} \\ B_x - 3C_x &= 0 & & C_x = -2.5 \text{ kN-m}\end{aligned}\quad (3.34)$$

DISCUSSION:

Since the torsion and bar formulae are so similar, it may be expected that this solution (both process and final result) match with Example 2.6. The constraints and loading are identical (except for units), and since J and G are not relevant to this problem, they do not alter the numerical values of the solution. This problem works out exactly the same as the equivalent bar problem.

Example 3.5 (Statically Indeterminate Shafts).

PROBLEM STATEMENT:

Consider the shafts shown in Figure 3.12. A circular aluminum rod is encased in a steel shell. The free ends of the rods are fused together and loaded with 200 kN-m. Determine the moment the fixed support provides to each material. In the following, subscript a refers to the aluminum and subscript s refers to the steel.

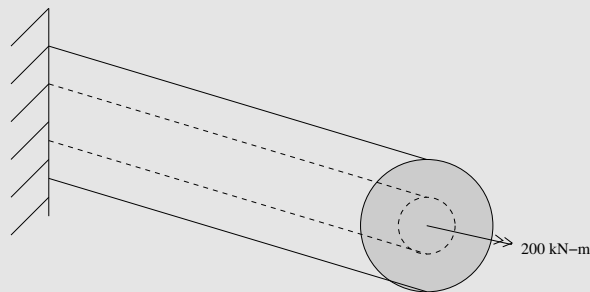


Figure 3.12: Example 3.5

KNOWNs:

- $D_a = 0.07 \text{ m} \rightarrow J_a = 2.357 \times 10^{-6} \text{ m}^4$
- $D_s = 0.08 \text{ m} \rightarrow J_s = 1.664 \times 10^{-6} \text{ m}^4$
- From Appendix B:
 - $G_a = 26 \times 10^9 \text{ N/m}^2$
 - $G_s = 77 \times 10^9 \text{ N/m}^2$
- Fixed support on left end for both materials
- Materials fused at right end
- Point load of 200 kN-m at right end

ASSUMPTIONs:

- Neglect end effects
- Members are long and slender

ANALYSIS:

Much like Example 2.7, we can work through this problem by considering the structure as two separate members with some interaction/constraint between them. Considering each rod individually, the requirements for *shaft analysis* are fully satisfied.

The free-body diagram is shown in Figure 3.13. Applying the equation of static equilibrium:

$$\sum M_{Ox} = 0 = M_s + M_a + 200 \quad (3.35)$$



Figure 3.13: Free-body diagram

Again, equilibrium is insufficient to solve for the reactions and a compatibility condition is necessary. Because the two shafts are fused at the free end, they deform together. Therefore, we can define the compatibility condition:

$$\phi_a = \phi_s \quad (3.36)$$

$$\int_L \frac{T_a}{J_a G_a} dx = \int_L \frac{T_s}{J_s G_s} dx$$

This compatibility condition allows us to solve the two shaft problems simultaneously. In this case each material is subjected to a single point load, so sections are not necessary. The internal torques can be expressed in terms of the reaction moments.

$$T_a = -M_a \quad T_s = -M_s \quad (3.37)$$

Once again, the integrals reduce to multiplication. Substituting the definitions of the internal torques into equation (3.36) and simplifying:

$$\frac{-M_a L}{J_a G_a} = \frac{-T_s L}{J_s G_s} \quad (3.38)$$

The lengths of the two shafts are equal and cancel. Substituting appropriate values of the shear modulus and polar second area moment into equation (3.38), we find a second equation to use with equilibrium:

$$M_s = 2.09M_a \quad (3.39)$$

Combining with equilibrium the reaction moments are found to be:

$$\begin{aligned} M_a &= -64.7 \text{ kN}\cdot\text{m} \\ M_s &= -135.3 \text{ kN}\cdot\text{m} \end{aligned} \quad (3.40)$$

DISCUSSION:

- Notice that unlike Examples 2.6 and 3.4, these results are not numerically equivalent to those in Example 2.7. This is because the values of G and J do not cancel in this problem as they did in the previous example; G and J are not the same as E and A , hence the difference between the examples.
- The moment born by the steel is more than twice the moment born by the aluminum, compared to Example 2.7 where the aluminum carried more total force. This is primarily due to the difference between J and A . Although they both measure the relevant geometric property of the member, they are physically and mathematically quite different. The area is independent of shape or distribution of the material, but the polar second area moment is specifically defined by the distance of the material from the torsional axis. Since the steel is much further from the axis than the aluminum, J_s is substantially greater than J_a (even though A_s is less than A_a).
- J measures the geometric stiffness of the shaft. Given a set amount of material, we can design shafts to be stiffer (lower stress and deformation for a given torque) by pushing as much of that mass away from the axis as possible. A tubular shaft is much stiffer than a solid shaft of the same mass.

3.4 Transmission of Power by Circular Shafts

Shafts are not always structural support elements. Often they are used to transmit motion and power; it is much easier spin a gear through 1000 revolutions than develop a piece of machinery that displaces a far enough linear distance to provide equivalent work.

Intuitively, transmission of power through a spinning shaft seems to be in violation of one of our most basic assumptions; that the structure be static. Recall how the condition of static equilibrium is derived. According to Newton's laws, if the net force (or moment) acting on an object is zero, then it is not accelerating. As long as the shaft has a constant angular velocity, the torques must still balance and equilibrium is still satisfied.

Total work due to rotation is defined as:

$$W = T\theta \quad (3.41)$$

where T is the applied torque and θ is the angle through which the body rotates (in radians). In the case of a power shaft, the rotation is continuous, so absolute work is not particularly meaningful. Much more useful in general is the rate of work, or **power**:

$$P = \frac{dW}{dt} = \frac{d(T\theta)}{dt} = T \frac{d\theta}{dt} = T\omega \quad (3.42)$$

Since the torque is constant, only the angle is differentiated, producing **angular**

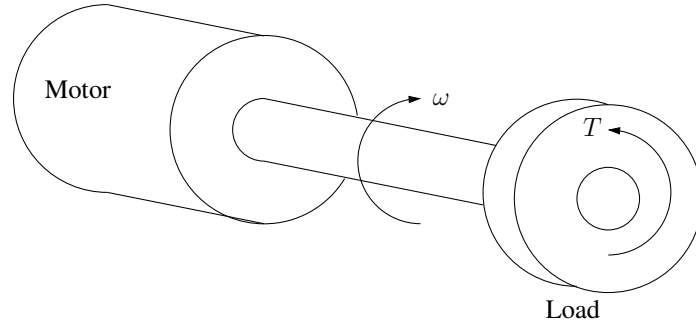


Figure 3.14: Motor motion and torque

velocity ω , measured in radians per unit time. Note in Figure 3.14 that the torque and angular velocity point opposite directions. This is because the torque applied to the shaft by its load will point in the opposite direction of the torque provided by the shaft. Power can be increased by increasing the torque or the angular velocity. Conversely, increasing the angular velocity will lower the required torque for a given power output.

Example 3.6 (Driveshaft).

PROBLEM STATEMENT:

Consider the same driveshaft as Example 3.2, part (b). If the engine provides 150 horsepower when the driveshaft is spinning at 1000 RPM and the maximum allowable shear stress in the material is 170 MPa, determine the minimum thickness of the shaft's wall.

KNOWNs:

- $R_o = 0.03$ m
- $L = 1.5$ m
- $M = 475$ N-m
- From Appendix B, $G = 26$ GPa
- Failure stress: $\tau = 170$ MPa
- $P = 150$ hp
- $\omega = 1000$ RPM

ASSUMPTIONs:

- Neglect end effects

ANALYSIS:

Since the shaft is spinning at a constant angular velocity, the choice to use *shaft analysis* described in Example 3.2 is still valid.

This problem is solved through straightforward application of equation (3.42), but care must be taken with units. Power must be converted to watts (joules per second) and angular velocity must be converted to radians per second.

$$\begin{aligned}
 P &= (150 \text{ hp}) \left(745.7 \frac{\text{W}}{\text{hp}} \right) = 111855 \text{ W} \\
 \omega &= (1000 \text{ RPM}) \left(2\pi \frac{\text{rad}}{\text{rev}} \right) \left(\frac{1 \text{ min}}{60 \text{ sec}} \right) = 104.7 \text{ rad/s}
 \end{aligned}
 \tag{3.43}$$

Substituting these values into the power equation and solving for the torque:

$$T = \frac{P}{\omega} = \frac{111855}{104.7} = 1068 \text{ N-m}
 \tag{3.44}$$

From the torsion formula:

$$J = \frac{TR}{\tau} = \frac{(1068)(0.03)}{170 \times 10^6} = 1.885 \times 10^{-7} \text{ m}^4 \quad (3.45)$$

Applying the definition of the polar second area moment, we can solve for the interior radius.

$$R_i = \left(R_o^4 - \frac{2J}{\pi} \right)^{1/4} = 0.0288 \text{ m} \quad (3.46)$$

DISCUSSION:

- The requirement that the shaft rotate at constant angular velocity is important for two primary reasons:
 - There is no acceleration, so torques balance.
 - The motor power or torque may independently vary with speed (as well as other factors such as fuel usage, current, etc.). When plotted against engine speed, these may be nonlinear curves called **torque and power curves**. If the speed changes with respect to time, then so does the torque and the simplification of Equation (3.42) is not valid.
- For the angular velocity to be constant, the first law of thermodynamics requires that the power provided by the motor be ‘consumed’ by the loads. That is, for power transmission problems we can say that the net power is zero. This follows logically from Equation (3.42); if the net torque is zero, then the net power must be as well.

3.4.1 Strain Energy

Closely related to the discussion of power transmission is the strain energy of the shaft. The definition of strain energy density remains the same as presented in equation (1.83). It is the integral of the stress with respect to its corresponding strain. In the case of a torsion problem, the stress and strain are shear.

$$u = \int \tau d\gamma \quad (3.47)$$

Substituting Hooke’s law for shear into the definition of strain energy density:

$$u = \int G\gamma d\gamma = \frac{G\gamma^2}{2} = \frac{\tau^2}{2G} = \frac{1}{2}\tau\gamma \quad (3.48)$$

Once again, we see the clear analogy between shafts (3.48) and bars (2.51). The stress at a point in the shaft is defined by equation (3.13), and shear strain by

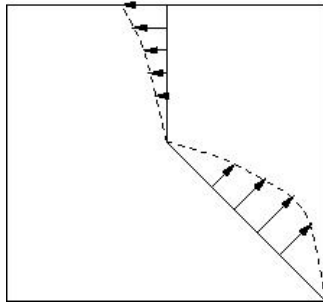
(3.4). Integration over the volume produces the total strain energy.

It is also possible to derive the strain energy using external work (3.41) and the first law of thermodynamics, although this is perhaps less general.

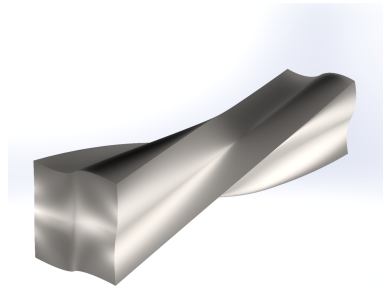
3.5 Thin-Walled Tubes*

In general, analysis of torsion for non-circular shafts is quite complicated. A circle is axisymmetric; any feature is independent of the angle of orientation within the circle. The shear stress is identical along every radial line. It varies linearly from the center of the cross-section to the outer radius. The torque affects the cross-section uniformly. None of this is true for a non-circular shaft. In a given cross-section, the torque is distributed irregularly.

Consider a square prismatic shaft. The corners of the cross-section are further from the centerline than the midpoint of each edge. The stresses are not linear, vary both radially and angularly, and are not necessarily maximized with distance from the axis. Figure 3.15(a) shows a simple illustration of these shear stresses along two radial lines. These effects culminate in **warping** of the shaft, creating three-dimensional deformation that changes the shape of the shaft Figure 3.15(b). Close inspection of the exposed face of the shaft in Figure 3.15(b) shows that the irregular shear stresses even cause the cross-sections to deform out of plane.



(a) Stress in a square cross-section



(b) A square torsion member, deformed

Figure 3.15: Torsion in a non-circular shaft

Fortunately, we can perform some analysis on certain non-circular shafts, if the proper conditions are met. Consider a *closed* prismatic tube of arbitrary shape, as shown in Figure 3.16. For a non-circular cross-section, the shear stress may vary along the perimeter. If the tube is *thin-walled*, then the radial variable does not change significantly from the inner surface to the outer, and τ can be treated independently of the depth of the material. This is counter to what we have for circular cross-sections, where the shear stress is constant around a circumference and varies with radius. A tube is considered *thin-walled* when the thickness of the tube is much less than the width of the cross-section. The thickness is not required to be constant around the perimeter of the tube, so in

general the greatest thickness must be much less than the smallest cross-section width.

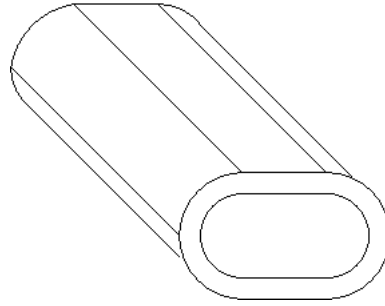


Figure 3.16: A thin-walled tube

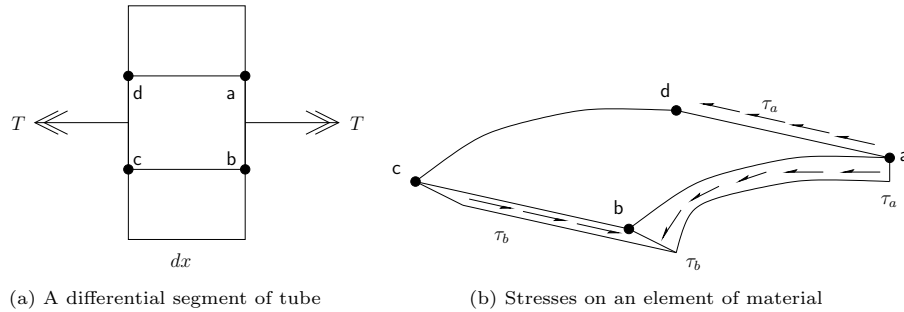


Figure 3.17: Stresses in thin-walled tubes

To evaluate the state of stress, we must consider a differential slice of the shaft (Figure 3.17(a)). Point *a* in general experiences a stress τ_a and point *b* a stress τ_b , with some variation along the perimeter between those points (Figure 3.17(b)). The thickness also varies from t_a to t_b along this curve. Shear stresses on orthogonal surfaces are equivalent, so τ_a and τ_b also act along the surfaces parallel to dx . The stress does not vary along the length (for a differential element), so stress is constant on surfaces *ad* and *bc*.

The surfaces *ad* and *bc* are rectangular, and the shear stresses are approximately constant along each. With this in mind, we can determine the total force acting on each of those faces:

$$\begin{aligned} F_{ad} &= \tau_a A_{ad} = \tau_a (t_a dx) \\ F_{bc} &= \tau_b A_{bc} = \tau_b (t_b dx) \end{aligned} \quad (3.49)$$

Note that due to the symmetry of shear stresses, these two forces point in opposite directions. From static equilibrium, we have:

$$F_{ad} = F_{bc} \quad (3.50)$$

Substituting equation (3.50) into (3.49) and simplifying:

$$\tau_a t_a = \tau_b t_b \quad (3.51)$$

Recall that this section of material is completely arbitrary, so the relation (3.51) holds for any points in the cross-section. Therefore, we have:

$$q = \tau t = \text{constant} \quad (3.52)$$

The quantity q is known as the **shear flow**. Shear flow is useful for the analysis of many thin-walled structures (known as **monocoques** or **semi-monocoques**), not just torsion. In such structures, the product of the shear stress and the thickness at a point is often more informative than the stress itself. The shear flow is sometimes also denoted f .

The shear flow gets its name from similar behavior in fluids flowing through certain channels. From equation (3.52) it is clear that shear stress will be higher in thinner regions of the tube than thicker. In that sense, the stress could be said to be “flowing” through the material; at a small thickness, more stress is needed to maintain the same shear flow. Shear flow is commonly illustrated in this way, as in Figure 3.18.

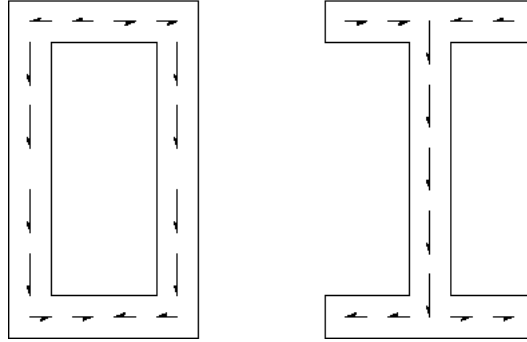


Figure 3.18: Shear flow in thin-walled sections

If we can relate the torque to the shear flow, then it is possible to determine the shear stress at every location in the shaft. Consider a cross-section of the shaft. Because the stress is independent of the radial distance through the thickness, we define a single **median line** equidistant from the inner and outer surfaces, shown as the dashed line in Figure 3.19. Any point quantities, such as forces, will be considered on the median line. The total length of the median line p_m (slightly less than the outer perimeter, slightly greater than the inner) is parametrized by a coordinate s . Since the tube is closed, $s = 0$ and $s = p_m$ are the same point. The origin of this coordinate, $s = 0$, is an arbitrarily selected reference point.

Consider a differential segment of the median line, ds . Because this segment is so small, it is effectively straight and the thickness over the segment is

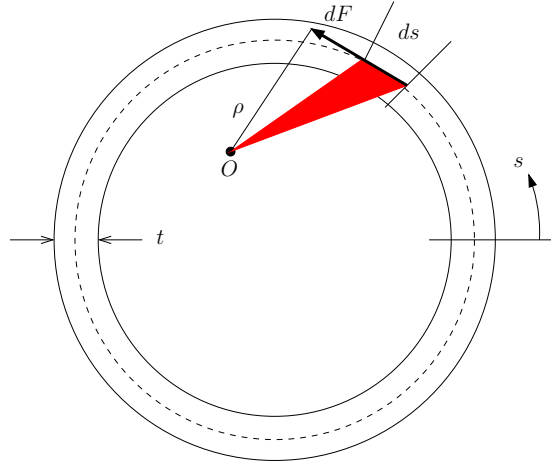


Figure 3.19: Tube cross-section

constant. Therefore, the shear force acting on this segment is:

$$dF = \tau dA = \tau(t ds) = q ds \quad (3.53)$$

The moment of this force about any point O is determined by multiplying it by the *perpendicular* distance ρ (not necessarily the total distance to the point force).

$$dM = \rho dF = \rho q ds \quad (3.54)$$

The total moment on the cross-section, which is equal to the internal torque at that location, is found by summing all differential moments along the median line:

$$T = \int dM = \oint \rho q ds = q \int_0^{p_m} \rho ds \quad (3.55)$$

Since shear flow is a constant along s (unlike τ or t), it may be pulled out of the integral. The integral along the closed contour s reduces to integration over the perimeter. The perpendicular distance ρ is a nontrivial function of the coordinate s . Determination of its precise functional nature requires detailed geometric information of the median line, the normal and tangent vectors at each point, and nonlinear (trigonometric) relations. It is generally much simpler to make use of a geometric interpretation of the terms in the integral.

As shown in Figure 3.19, a is the triangle formed by connecting the edges of ds and point O . The area of this triangle, shown as the shaded region in Figure 3.19, is given as:

$$dA_m = \frac{1}{2}bh = \frac{\rho ds}{2} \quad (3.56)$$

The base of the triangle is ds and the height is ρ . Rearranging (3.57):

$$\rho ds = 2dA_m \quad (3.57)$$

Substituting this relation into equation (3.55), the torque is defined as:

$$T = q \int_0^{p_m} 2dA_m = 2qA_m \quad (3.58)$$

The differential area dA_m is the area of a triangle swept out by a line connecting O to the median line. Therefore, A_m is the *entire area enclosed by the median line* (not the cross-sectional area of the tube). Substituting the definition of shear flow into (3.58) and rearranging produces the torsion formula for thin-walled tubes:

$$\tau = \frac{T}{2tA_m} \quad (3.59)$$

It is now possible to determine the shear stress at any point in the tube. Note that since thickness is in the denominator, and A_m and T are constant for a given cross-section, then the maximum shear stress occurs at the minimum thickness. The polar second area moment is not present in this version of the torsion formula; the geometric stiffness is defined by the median area and the thickness.

Several critical assumptions were made in developing equation (3.59), and if any of them are violated then we can expect significant errors in our results.

1. The shaft is thin-walled. All analysis is reduced to the median line.
2. The shaft is a closed tube. An open section can weaken the structure and could introduce modes of deformation other than torsion.
3. The shaft is prismatic. Shape and thickness do not vary along the length.

It is possible to develop an expression for the angle of twist for a thin-walled tube using the strain energy density. However, due to the additional deformation in non-circular shafts, this is less accurate and more restrictive than might be generally useful.

Example 3.7 (Circular Tubes).

PROBLEM STATEMENT:

Consider a circular tube subjected to a 100 kN-m torque. The outer radius is 0.1 m. Compare the stress in the tube computed using circular shaft theory and thin-walled tube theory if (a) $R_i = 0.095$ m and (b) $R_i = 0.05$ m.

KNOWN:

- $T = 10^5$ N-m

- $R_o = 0.1$ m
- For part (a):
 - $R_i = 0.095$ m
 - $t = 0.005$ m
- For part (b):
 - $R_i = 0.05$ m
 - $t = 0.05$ m

ASSUMPTIONS:

- Shaft is long and slender
- No end effects
- Small deformation
- Linearly elastic material

ANALYSIS:

Part (a)

This circular tube has simple geometry, with constant thickness around the perimeter. The distance ρ is equivalent to the median radius r_m .

$$\begin{aligned} r_m &= \frac{R_o + R_i}{2} = 0.0975 \text{ m} \\ A_m &= \pi r_m^2 = 0.0299 \text{ m}^2 \end{aligned} \quad (3.60)$$

The shear stress for a thin-walled tube is given by Equation (3.59).

$$\tau_{tw} = \frac{T}{2tA_m} = \frac{10^5}{2(0.005)(0.0299)} = 334 \text{ MPa} \quad (3.61)$$

To analyse this as a classical shaft, the polar second area moment is necessary.

$$J = \frac{\pi(R_o^4 - R_i^4)}{2} = 2.914 \times 10^{-5} \text{ m}^4 \quad (3.62)$$

From Equation (3.13) we can determine the shear stress.

$$\tau_{cl} = \frac{Tr}{J} = \left\{ \begin{array}{ll} 326 \text{ MPa} & r = R_i \\ 335 \text{ MPa} & r = r_m \\ 343 \text{ MPa} & r = R_o \end{array} \right\} \quad (3.63)$$

Part (b)

If the inner radius decreases, the thickness increases. Repeating the calculations from Part (a):

$$r_m = \frac{R_o + R_i}{2} = 0.075 \text{ m} \quad (3.64)$$

$$A_m = \pi r_m^2 = 0.0177 \text{ m}^2$$

$$\tau_{tw} = \frac{T}{2tA_m} = \frac{10^5}{2(0.05)(0.0177)} = 56.5 \text{ MPa} \quad (3.65)$$

$$J = \frac{\pi(R_o^4 - R_i^4)}{2} = 1.473 \times 10^{-4} \text{ m}^4 \quad (3.66)$$

$$\tau_{cl} = \frac{Tr}{J} = \left\{ \begin{array}{ll} 34.0 \text{ MPa} & r = R_i \\ 50.9 \text{ MPa} & r = r_m \\ 67.9 \text{ MPa} & r = R_o \end{array} \right\} \quad (3.67)$$

DISCUSSION:

There are clear differences between the classic torsion formula for circular shafts and the approximation for thin-walled tubes.

- The thin-walled approximation provides a single value of the stress, while the classic theory allows stress to vary through the thickness.
 - For the shaft in part (a), the variation in stress from the inner surface to the outer surface is about 6%.
 - For the shaft in part (b), the variation of stress through the thickness is as much as 50%.
 - If the shaft satisfies the thin-wall requirement, as in part (a), the variation in the stress is not significant.
- The value of the stress from the thin-walled approximation is much more accurate when $t \ll r_m$.
 - For the shaft in part (a), the greatest difference between the stress calculated using classic theory and the stress from the thin-walled approximation is about 2.5%.
 - For the shaft in part (b), the greatest difference between the stress calculated using classic theory and the stress from the thin-walled approximation is about 20%.
 - The error reduces as the relative thickness (t/r_m) approaches zero.
- In this problem, thickness (and therefore stress) is constant around the perimeter. In general, the thickness may vary, requiring identification of locations with peak stress.

3.6 Chapter Summary

Shafts are the second surrogate model typically considered in solid mechanics. Due to extensive similarities with bar models, the analysis of shafts is straightforward and relatively easy. Again, this model is only applicable when the following assumptions and restrictions hold:

1. The material is continuous, homogeneous, isotropic, and linearly elastic
2. The structure is long, slender, and straight
3. The member undergoes only small deformation
4. The structure is loaded only by moments oriented along its longitudinal axis
5. The torques are approximately centered on the axis of the member
6. Area changes occur gradually
7. The shaft is circular
8. Certain non-circular shafts may be analyzed if the following additional assumptions hold:
 - (a) The shaft is a thin-walled tube ($t \ll D$)
 - (b) The tube is prismatic
 - (c) The tube has a closed section

As with bars, the mechanical analysis of the structure reduces to a purely one-dimensional problem. The bar twists about its longitudinal axis, resulting in shear strain. Following moment equilibrium, the shear stress is related to the internal torque in the shaft. Through Hooke's law, the torque is directly related to shear strain and angle of twist.

Angle of twist can be determined for specific segments of the shaft or the entire object. A useful notation for describing such deformation $\phi_{B/A}$ explicitly defines the motion of point B relative to point A , analogous to bars. This concept of relative motion is particularly useful in determining the compatibility conditions necessary for analysis of statically indeterminate shafts.

Transmission of power by rotating shafts, perhaps the most common use of circular shafts, and energy effects are also explored. The strain energy and strain energy density are measures of work stored in the material. In the case of shafts, the force component of the work is defined by the shear stress and the motion is in terms of shear strain. With proper substitutions, the energy relation can be put in terms of torque and angle of twist.

If a shaft is non-circular, then torsion produces complex, three-dimensional deformation and erratic stress fields. However, if the structure is a thin-walled, closed tube, then stress is approximately constant through the material thickness and analysis is simplified to a single radial location. It is possible to determine

the stress along the median line using the concept of shear flow. Deformation of a thin-walled tube is still quite complicated, but good approximations of stresses are possible.

3.6.1 List of Symbols

τ :	Shear stress
γ :	shear strain
T :	Internal torque
R :	Radius of the shaft
r :	Radial coordinate
ϕ :	Angle of twist of the shaft
$\phi_{b/a}$:	Motion of point b relative to point a
G :	Shear modulus
J :	Polar second area moment
L :	Undeformed length of shaft segment
W :	Torsional work
P :	Power transmitted
ω :	Angular velocity
u :	Strain energy density
q :	Shear flow
t :	Tube wall thickness
A_m :	Median area

Chapter 4

Beam Analysis

Bars and shafts are quite similar in that they are both slim, straight members subjected to axial loads. In both cases, the models reduce to one-dimensional representations of the physics. Action occurs along the longitudinal axis, with loads and deformation both parallel to the axis (in the vector sense). In this chapter, we begin investigating more advanced models by introducing *transverse* deformation. If certain requirements are met while doing so, however, we can retain a **geometrically one-dimensional** model, similar to bars and shafts; that is, all behavior is defined along a single axis or curve. When such a structure is subjected to an off-axis load, it bends. We call these members **beams**.

4.1 Fundamentals of Beam Models

Structurally, a beam is similar to bars and shafts. For our purposes, the following define a beam:

1. The structure is loaded by moments about a single lateral axis (the z -axis of Figure 4.1)
2. The structure is long, slender, and straight
3. Area changes occur gradually
4. The beam cross-section is symmetric about the y -axis, as shown in Figure 4.1

The requirement that the beam be long and slender is primarily responsible for allowing us to reduce this to a 1D problem. The beam is represented by a single line (not necessarily the center line). Strictly speaking, the beam does not have to be straight. Moments do not need to lie on a single line to balance. However, analysis of curved beams is rather more advanced than the material presented here, so our study will be restricted to straight members. Furthermore, moments that do not lie on the curve defining our 1D problem may cause

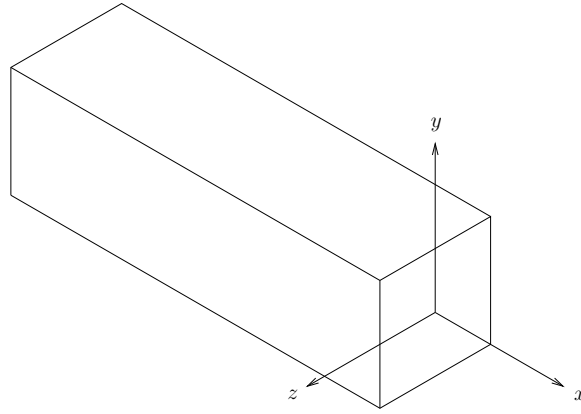


Figure 4.1: Beam coordinate system

local non-uniformities, but by applying Saint-Venant's principle, these are neglected.

The major difference in a beam model from the previous surrogate models discussed is its deformation mode. The beam bends about the z -axis, deflecting away from the longitudinal axis. The requirement of symmetry in the cross-section is important here. A load may result in additional modes of deformation (such as twisting or warping) if applied to a non-symmetric cross-section. These effects might be neglected as local according to Saint-Venant's principle, but for the time being this issue will be avoided entirely.

4.1.1 Vocabulary and Beam Anatomy

It is important to clarify certain terms with regards to these beams. The x -dimension of the beam is called the **length**. The y -dimension is often referred to as **depth**, although it may sometimes be called **height**. The z -dimension is known as the **width**. It may rarely be called **thickness**, but this should be avoided as that may refer to other, more specific dimensions. Width and depth typically refer to total dimensions of a beam.

Beams are very common structural elements, and some specific shapes of beams are most often used. These beams are typically formed with multiple elements. Vertical components of the beam are called the **web**, while horizontal components are known as **flanges**. Thickness often refers to the smaller dimension of each component. Figure 4.2 shows several common structural beam shapes.

- I-beam: The most common beam shape. A single web separating two flanges, centered on the web. Flanges may be the same size (symmetric I-beam) or different sizes (asymmetric I-beam).
- H-beam: An I-beam oriented such that the web and flanges change roles. Two webs separated by a single flange.

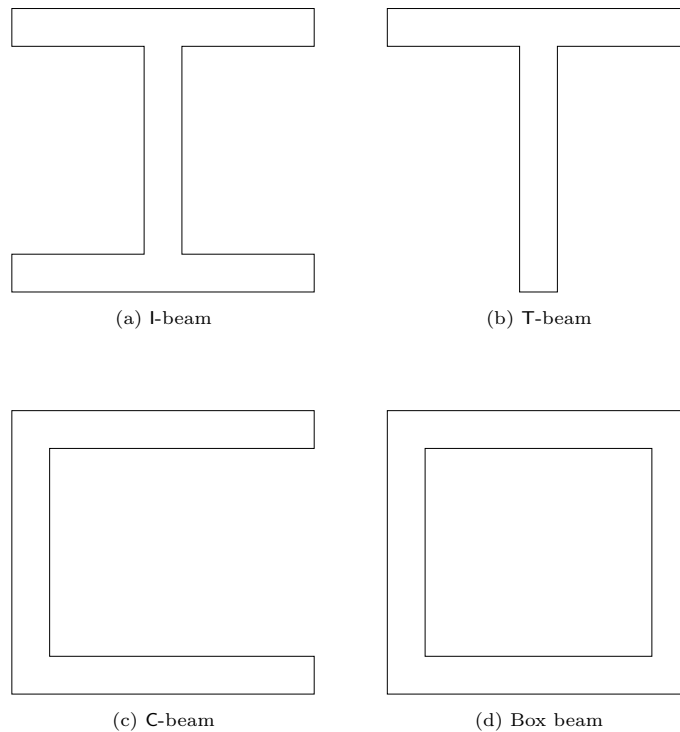


Figure 4.2: Common types of beams

- T-beam: A web with a single flange. Used more rarely than I-beams, but may be useful in specific cases. The flange may be made of a brittle material (such as concrete) to bear large compressive loads, while the web could be a more ductile material capable of bearing tensile and shear loads.
- C-beam or channel beam: Similar to an I-beam, but the web is located on one side of the flanges, rather than centered. If rotated like an H-beam, it becomes symmetric.
- Box beam: A closed channel. Not as efficient as an I-beam for bending loads, but more capable of supporting torsion and shear loads.

4.2 Bending of Beams

Consider a beam of length L subjected to a point moment M_z at each end. This situation is called **pure bending**. The moments are such that the beam bends upward, as shown in Figure 4.3. Notice that the inside surface of the bent beam (the top in this case) has shortened while the outside surface (the bottom) has lengthened from the undeformed state. Intuitively, there must be some point between the two surfaces that has not changed length at all. This

plane is the **neutral surface**. When viewed from the side, it appears as a single line; we reduce our geometric description to this line. The intersection of the neutral surface with a cross-section of the beam is the **neutral axis**.

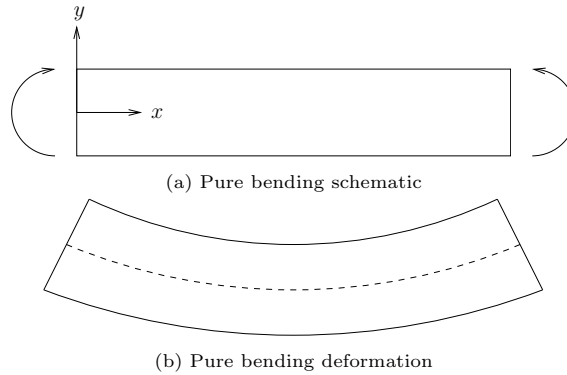


Figure 4.3: Pure bending behavior

Unlike bars and shafts, a deformed beam changes shape. This deformation is measured as **curvature**, the deviation of the shape from a straight line. Consider the beam shown in Figure 4.4. Consider two points on the curve. Normal rays drawn from these points will intersect at some third point. If the two points are close, say separated by an infinitesimal distance ds , then the rays will have the same length. This length ρ is called the **radius of curvature**. The radius of curvature is the radius of a circle that best approximates the curve at a point (such a circle is called an **osculating circle**). Thus, ds is a circular arclength. The arclength and radius are related through the angle of the section swept out, $d\theta$:

$$ds = \rho d\theta \quad (4.1)$$

Or,

$$\rho = \frac{ds}{d\theta} \quad (4.2)$$

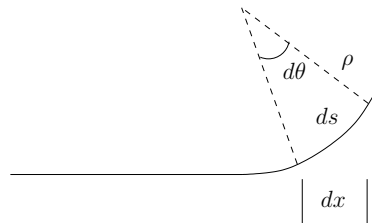


Figure 4.4: Curvature of a beam

The curvature κ is the reciprocal of the radius of curvature:

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds} \quad (4.3)$$

Intuitively, equation (4.3) describes the rate of change of orientation of the normal ray along the length of the curve. A higher curvature will change orientation more quickly; that is, a tighter curve has higher curvature.

If the beam undergoes only small deformation (Figure 4.4 shows highly exaggerated deformation) then the arclength ds is approximately equivalent to its horizontal projection dx . With this substitution, equation (4.3) reduces to:

$$\kappa = \frac{d\theta}{dx} \quad (4.4)$$

The curvature is now defined at every point along the length of the beam. Since angle θ is positive counterclockwise (in a right-hand coordinate system), curvature is positive if the beam curves upward. Similarly, a beam that curves downward has negative curvature, as illustrated in Figure 4.5.



Figure 4.5: Curvature sign convention

When a beam curves, the orientation of the cross-section planes changes. Note that in the case of pure bending, there are no shear forces on the yz -face of a cross-section. Therefore, there are no shear stresses τ_{xy} or strains γ_{xy} ; the shape of the cross-section does not change in that direction. In pure bending, beam cross-sections remain planar and normal to the top, bottom, and neutral surfaces of the beam (see Figure 4.6)! In fact, each cross-section aligns with the radius of curvature at that point. While cross-sections may not deform with respect to the x -axis, deformation in the yz -plane is entirely possible. If the top of the beam in Figure 4.6 reduces in length (along the x -axis), then Poisson effects require an increase in width of the cross-section at the top. Likewise, if the bottom surface of the beam elongates, then the width of the cross-section at that point must decrease, resulting in the keystone shape of Figure 4.6(b). This change of shape (or more specifically, change of angle between perpendicular lines) creates a shear strain γ_{yz} . This shear strain is not typically critical for beam analysis, however. For the remainder of this chapter, focus will be on γ_{xy} .

The curvature shown in Figure 4.4 is that of the neutral surface. The line has become curved, but has not changed length. The normal strain along the neutral surface $\varepsilon_{xx} = 0$. Therefore the stress is also zero. As we move away from the neutral surface, nonzero normal stresses and strains develop. Consider Figure 4.7. Again, the curved neutral surface is shown (exaggerated for visibility), with

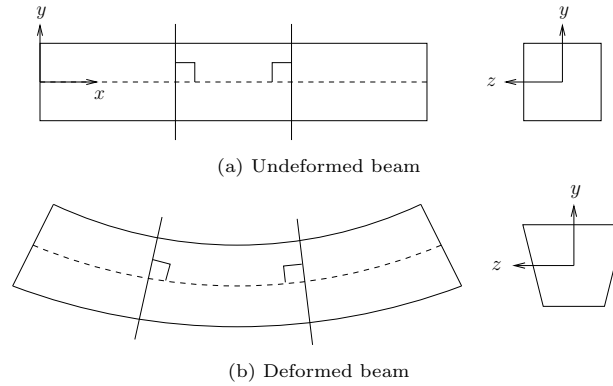


Figure 4.6: Cross-section behavior under pure bending

a given radius of curvature ρ . At some distance y from the neutral axis, the arclength a is less than that at the neutral surface.

$$\begin{aligned} a &= (\rho - y)d\theta \\ &= \rho d\theta - yd\theta \end{aligned} \quad (4.5)$$

Assuming small deflection and substituting the definition of curvature (4.4) into Equation (4.5):

$$a = dx - \frac{y}{\rho}dx \quad (4.6)$$

Recall the definition of normal strain as change of length divided by original length (where the undeformed length is dx):

$$\varepsilon_{xx} = \frac{a - dx}{dx} = \frac{(dx - \frac{y}{\rho}dx) - dx}{dx} = -\frac{y}{\rho} \quad (4.7)$$

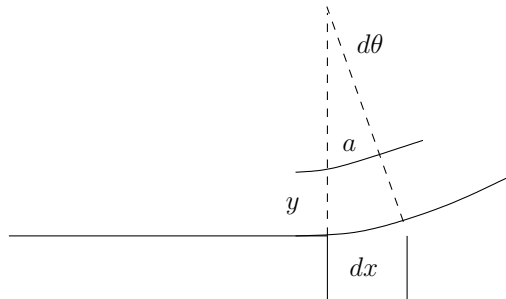


Figure 4.7: Longitudinal strain in pure bending

The radii of curvature at any distance y in the cross-section are collinear. This is a direct consequence of pure bending and the cross-sections remaining

normal to the neutral surface. It is reflected in equation (4.7); the longitudinal strain is linear with respect to distance from the neutral axis y . Cross-sections rotate rigidly with respect to each other. The result of equation (4.7) matches what we observe in beam deformation. In a beam with positive curvature, positive y locations would have negative longitudinal strains. Furthermore, if the y coordinate measures distance from the neutral axis in a cross-section, then the z -axis is the neutral axis.

One interesting and important point to note is that the longitudinal strain ε_{xx} is the only strain directly caused by the loading. All other strains develop as a result of the Poisson effect. Even though deformation in the beam is primarily transverse (y) deflection and shape change, the moment is tied to elongation in the x -direction.

4.3 Normal Stress in Beams

As mentioned previously, the transverse longitudinal strains ε_{yy} and ε_{zz} are due to the Poisson effect. Therefore, the stress field in pure bending is in fact one-dimensional. Assuming the material is linearly elastic, the 1D Hooke's law can be applied.

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(-\frac{y}{\rho}\right) \quad (4.8)$$

Stress also varies linearly with y , as illustrated in Figure 4.8. Since curvature can vary along the length of the beam, the normal stress is a function of x and y . The longitudinal stress σ_{xx} is often called **bending stress**.

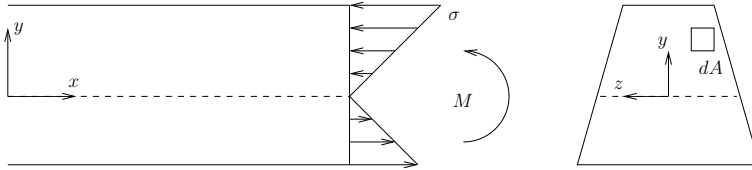


Figure 4.8: Bending stress in pure bending

There are two primary difficulties in using equation (4.8): (i) curvature may be difficult to measure and use in practical problems, and (ii) the distance y from the neutral surface requires determination of the location of the neutral surface. Both of these points will be addressed in the following.

The bending stress is normal to the cross-section. This cross-section is static, so the sum of forces acting on the section must be zero. Since the stress varies with y , the total force is not simply stress multiplied by the area. For a small enough portion of the cross-section, such as the differential area such as that shown in Figure 4.8, this is the case. The total force is therefore the sum of all forces acting on differential areas:

$$\sum F_x = 0 = \int \sigma_{xx} dA \quad (4.9)$$

From equation (4.8), the stress is known in terms of y .

$$\int \sigma_{xx} dA = \int \left(-E \frac{y}{\rho} \right) dA = 0 \quad (4.10)$$

Within a cross-section both E and ρ are constant, so the equilibrium condition reduces to:

$$\int y dA = 0 \quad (4.11)$$

The integral in equation (4.11) is recognizable from equation (1.43) as the first area moment about the neutral axis (A is the yz -plane in this case). Consider the definition of the centroid from Equation (1.36), and rearrange:

$$\bar{y}A = \int y dA \quad (4.12)$$

Substituting this relation into (4.11), it is clear that equation (4.11) only holds if the origin of the coordinate system is the centroid of the cross-section ($\bar{y} = 0$). In other words, *the neutral axis passes through the centroid of each cross-section!*

With the neutral axis/surface now fully-defined, it is possible to consider the bending moment directly. Recall that a moment is the product of a force and the perpendicular distance to the point considered. Following the previous discussion, the force is stress acting on a differential area. For a moment about the neutral axis, the perpendicular distance is measured by y (see Figure 4.8).

$$dM = y \cdot (-\sigma_{xx} dA) \quad (4.13)$$

Positive stress is pointed outward from the face, but a positive moment is counterclockwise about the positive z -axis, hence the negative sign in equation (4.13). Summing the differential moments over the area yields the total bending moment.

$$M = - \int y \sigma_{xx} dA \quad (4.14)$$

M is the internal bending moment acting on the cross-section. Substituting the definition of bending stress from equation (4.8):

$$M = \frac{E}{\rho} \int y^2 dA \quad (4.15)$$

The quantity $\int y^2 dA$ is the the second area moment of the cross-section with respect to the z -axis. Following the discussion in Section 1.1.5.2, this is completely deterministic from the geometry of the beam, and is treated as a known value I_{zz} , or simply I .

$$M = \frac{EI}{\rho} \quad (4.16)$$

Since E and I are positive constants for a cross-section, equation (4.16) requires that the internal moment and the curvature have the same sign. This

will dictate how internal bending moments are defined (in a following section). Substituting the radius of curvature from equation (4.7), the moment is related to longitudinal strain.

$$M = \frac{EI}{-y/\varepsilon_{xx}} \quad (4.17)$$

Finally, this equation may be rearranged or used with Hooke's law to determine longitudinal strain and stress in terms of the internal bending moment.

$$\varepsilon_{xx} = -\frac{My}{EI} \quad (4.18)$$

$$\sigma_{xx} = -\frac{My}{I} \quad (4.19)$$

Equations (4.18) and (4.19) define the response of the beam to applied loads. As expected from our previous observations, the bending stress and strain are proportional to distance from the neutral axis. The magnitudes reach a maximum value at the furthest distance from the neutral axis, and are zero at the neutral axis. Equation (4.19) is sometimes called the **flexure formula**. This also matches perfectly with the observation that cross-sections do not deform or rotate relative to the neutral axis.

Use of these formulae do require determination of the second area moment, discussed in detail in Section 1.1.5.2. Recall that the second area moment measures the distance of material from the axis. From Equation (4.19), a greater value of I will result in lower stress. Pushing material further from the axis increases the stiffness of the beam. This is why structural elements such as I-beams are so common; it is a much more efficient use of the material.

4.3.1 Assumptions

It is important at this point to discuss some of the assumptions and their importance to developing this beam model:

1. Symmetric cross-section: This assumption was in play throughout these derivations with regards to deformation modes, bending axes, etc.
2. Small deformation: This allowed us to simplify the curvature to horizontal length in equation (4.4).
3. Pure bending: The absence of shear forces along the cross-section is necessary for the cross-sections to remain undeformed and normal to the neutral surface.
 - This cross-section behavior allows for the relation of strain ε_{xx} to curvature in (4.7).
4. Homogeneous material: We treated E as a constant and were able to remove it from integrals in equations (4.10) and (4.15).

5. Static object: We used static equilibrium to define the location of the neutral axis in equation (4.10).
6. Gradual area change: Strictly speaking, these results hold for prismatic beams. If the area changes along the length, the resultant stresses will not be perfectly normal to a cross-section (see Section 2.1 for similar discussion on bars). Error in the results in such cases is rather low. The beam may have up to a 15° taper along its length before error in stress exceeds 5%.

Example 4.1 (Curvature).

PROBLEM STATEMENT:

Consider a wire wrapped partly around a drum, as shown in Figure 4.9. The drum and wire radii are known. Determine (a) the maximum strain in the wire and (b) the moment required to bend the wire around the drum.

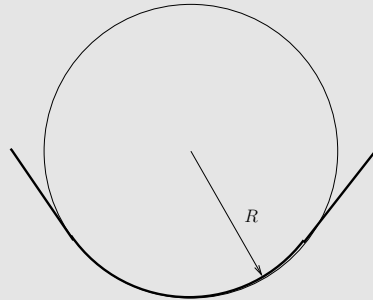


Figure 4.9: Example 4.1

KNOWNs:

- Drum radius $R = 1$ m
- Wire diameter $d = 1$ mm
- Steel wire, $E = 200$ GPa (from Appendix B)

ASSUMPTIONs:

- The wire is in a state of pure bending
- The drum is rigid relative to the wire
- The wire is wound tightly against the drum

ANALYSIS:**Part (a)**

The diameter of the wire is much less than the circumference of the drum, so it may be considered long and slender. With the given assumptions, this is suitable for *beam analysis*.

The strain in the wire can be related to the curvature through equation (4.7). Since the drum is circular and the wire is snug against it, the wire is bent into a circle. The radius of curvature of a circle is the same as the radius of the circle; in this case, the radius of the bent neutral surface of the wire. Because the wire is symmetric, the centroid (and therefore the neutral surface) is located at the center.

$$\rho = R + d/2 = 1.0005 \text{ m} \quad (4.20)$$

The curvature is positive because the wire is bent upward in Figure 4.9. The maximum strain in the wire occurs at the peak distance from the neutral surface. Substituting values into equation (4.7):

$$\varepsilon_{xx} = -\frac{y_{max}}{\rho} = -\frac{d/2}{\rho} = -\frac{0.0005}{1.0005} = -0.0005 \quad (4.21)$$

This strain is compressive. The result is logical because this value of y is against the surface of the drum. Since the wire is symmetric, we could alternatively have chosen $y = -0.0005 \text{ m}$ and found an equivalent tensile strain.

Part (b)

To determine the required bending moment (with the assumption of pure bending), Equation (4.18) may be used. First however, it is necessary to find the second area moment of the wire's cross-section with respect to the neutral axis. For a circle, I may be found from equation (1.46) or Appendix A.

$$I = \frac{\pi d^4}{64} = 4.91 \times 10^{-14} \text{ m}^4 \quad (4.22)$$

The strain at the outer surface of the wire is known from equation (4.21). Rearranging and substituting values into equation (4.18):

$$M = -\frac{EI\varepsilon_{xx}}{y} = -\frac{(200 \times 10^9)(4.91 \times 10^{-14})(-0.0005)}{0.0005} = 0.01 \text{ N}\cdot\text{m} \quad (4.23)$$

DISCUSSION:

We can confirm after solving the problem that the beam is undergoing small deformation. The value of the strain determined from equation (4.21) confirms this. Furthermore, the radius of the drum is much greater than the diameter of the wire, indicating a low curvature for the problem.

The value of I may seem suspect, perhaps preposterously low, but that is an artifact of the units used. The diameter of the wire is 10^{-3} m, and the second area moment requires the 4th power of that. Intuitively, such a skinny wire would not have much stiffness, which is also reflected in the value of the moment computed in (4.23).

Example 4.2 (Pure Bending).

PROBLEM STATEMENT:

Consider two beams, cross-sections shown in Figure 4.10. Determine the peak stress in each beam if they are subjected to a 1 kN-m moment.

KNOWNs:

- Aluminum, $E = 70$ GPa (from Appendix B)
- For beam **A**:
 - Width $b = 3$ cm
 - Depth $h = 4$ cm
- For beam **B**:
 - Flange width $b = 3$ cm
 - Flange thickness $t = 1$ cm
 - Web height $h = 6$ cm
 - Web thickness $t = 1$ cm
- Bending moment $M = 1$ kN-m

ASSUMPTIONs:

- Pure bending
- Beam is long and slender

ANALYSIS:

Both shapes are symmetric about the y -axis, made from homogeneous, isotropic, linearly elastic material. With the assumptions, this problem satisfies the requirements for *beam analysis*. It will be necessary to check for small deformation after the analysis is complete.

The stress in each beam may be determined using the flexure formula (4.19). The neutral axis and second area moment of each beam must be determined. In this case, both beams are vertically symmetric, so the centroid is located directly in the center of each shape.

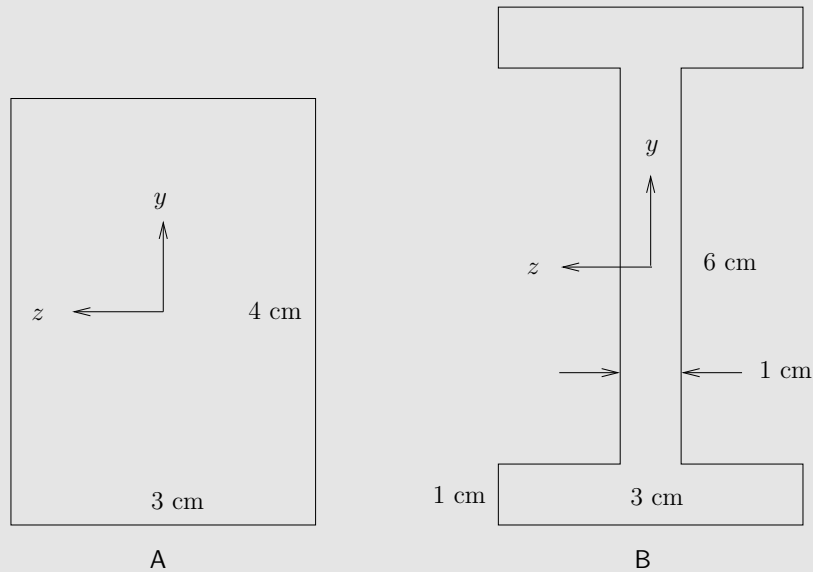


Figure 4.10: Example 4.2

From equation (1.46) or Appendix A, the second area moment of a solid rectangle is:

$$I_A = \frac{bh^3}{12} = \frac{(0.03)(0.04)^3}{12} = 1.6 \times 10^{-7} \text{ m}^4 \quad (4.24)$$

An I-beam is a composite shape, so determination of I_B requires the use of the parallel axis theorem, equation (1.48). Consider the flanges and web individually:

$$\begin{aligned} I_{flange} &= \frac{bt^3}{12} = \frac{(0.03)(0.01)^3}{12} = 2.5 \times 10^{-9} \text{ m}^4 \\ \tilde{y}_{flange} &= 0.005 \text{ m} \\ A_{flange} &= 0.0003 \text{ m}^2 \end{aligned} \quad (4.25)$$

$$\begin{aligned} I_{web} &= \frac{th^3}{12} = \frac{(0.01)(0.06)^3}{12} = 1.8 \times 10^{-7} \text{ m}^4 \\ \tilde{y}_{web} &= \tilde{y} = 0.04 \text{ m} \\ A_{web} &= 0.0006 \text{ m}^2 \end{aligned} \quad (4.26)$$

Substituting these values into equation (1.48):

$$\begin{aligned}
 I_B &= \sum (I_i + A_i d_i^2) \\
 &= (I_{web} + A_{web}(\bar{y} - \tilde{y}_{web})^2) + 2(I_{flange} + A_{flange}(\bar{y} - \tilde{y}_{flange})^2) \quad (4.27) \\
 &= (1.8 \times 10^{-7} + 0) + 2(2.5 \times 10^{-9} + 3.675 \times 10^{-7}) \\
 &= 9.2 \times 10^{-7} \text{ m}^4
 \end{aligned}$$

The peak stress occurs at the furthest point from the neutral axis. Since both these beams are symmetric, the maximum tensile and compressive stresses are equal magnitude. Substituting values into equation (4.19):

$$\sigma_A = -\frac{My_{max}}{I_A} = -\frac{(1000)(0.02)}{1.6 \times 10^{-7}} = 125 \text{ MPa} \quad (4.28)$$

$$\sigma_B = -\frac{My_{max}}{I_B} = -\frac{(1000)(0.04)}{9.2 \times 10^{-7}} = 43.5 \text{ MPa} \quad (4.29)$$

DISCUSSION:

The stress in beam B is approximately one third of that in beam A, despite having the same amount of material (same cross-sectional area). The reason is that more of that material in B was at a further distance from the neutral axis, thereby increasing the beam's ability to support a load. From equation (4.27), it is clear that contributions of the flanges to the total second area moment of the I-beam are significant. The distance terms in the parallel axis theorem dominate. The flanges bear the majority of the bending stress; the primary purpose of the web is to separate the flanges.

Direct application of Hooke's law to find the strain will let us know if the beam underwent small deformation, and whether this analysis was valid.

$$\epsilon = \frac{\sigma}{E} = \frac{125 \times 10^6}{70 \times 10^9} = 0.0018 \quad (4.30)$$

This value is low enough to consider the requirement satisfied.

4.3.2 More on Non-Prismatic Beams

As mentioned previously, the results here are for prismatic beams. If a beam is prismatic, then the geometric properties of the cross-section are constant with respect to the x -coordinate. The area, second area moment I , and distance from the neutral axis are not functions of x . If the normal stress varies along the length, it is only due to variation in the internal moment $M(x)$. Thus, for prismatic beams, the maximum stress occurs at the same location as the

maximum bending moment.

If a beam is non-prismatic (with gradual area change), then both I and y_{max} may vary along the length. In such cases, the maximum stress may occur at any point in the beam, not only where the maximum moment occurs. In fact, it is possible to design a beam such that every point x experiences the same maximum stress. If the maximum allowable stress in the beam is S_A (for example, yield stress with a factor of safety), then a beam designed such that $\sigma(x, y_{max}) = S_A$ for all values of x is called **fully-stressed**. This might be advantageous because all excess material has been removed; every cross-section bears as much stress as it can without any waste. In other words, the weight of the beam is minimized. Such a procedure in general may require advanced mathematical optimization techniques which are not presented here, but simple cases could be designed with basic calculus.

Fully-stressed beams are ideal for many applications, but are relatively rare in practice. Variable cross-section beams may be difficult and expensive to manufacture. Furthermore, a beam is only fully-stressed under its design conditions. If the physical loading or constraints are different at all from the model, then the beam will not be optimal. That being said, fully-stressed beams (or approximately fully-stressed) do still appear in weight-critical situations, such as aircraft.

4.4 Non-Uniform Bending

So far we have considered only pure bending, i.e., bending caused by pure moments. In practice, pure bending moments are rare. More often, a moment is created by a transverse force (oriented in the y -direction). For example, a beam supporting any weight will be loaded by forces normal to the neutral surface. At any distance along the length of the beam from these forces, there will be moments. Such problems are referred to as **non-uniform bending**.

At first, such a distinction may seem insignificant given that an internal moment is still generated. However, there are some important consequences that require attention. The force will locally create normal stresses σ_{yy} , but these vanish quickly with distance from the load. A transverse (y) force is parallel to the cross-section (yz -plane), so results in a shear force along that surface. A shear force acting over an area produces a shear stress, which in turn requires a shear strain γ_{xy} . Recall that shear strain is a measure of the change of angle between material lines. In other words, the cross-sections do not remain perpendicular to the neutral axis in non-uniform bending. As illustrated in Figure 4.11, the cross-section of a beam warps under shear forces. Unfortunately, all of the deformation and stress analysis of a beam under pure bending relied on rigidity of the cross-section, and *is therefore invalid in non-uniform bending*.

Not all is lost, though. Experience has shown that while equations (4.7), (4.18), and (4.19) do not strictly hold in non-uniform bending, their results may still closely approximate the normal stress and strain field in certain beams. If the shear force acting on a beam is constant along the length, then each cross-

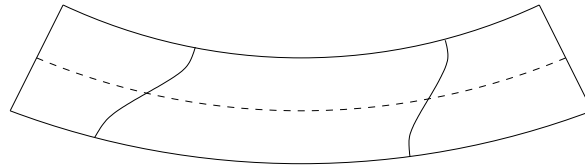


Figure 4.11: Non-uniform bending

section experiences the same shear strain and warp in the same manner. In that case, the longitudinal deformation (i.e., the normal strains) are unaffected by the warping. Even if the shear force varies continuously along the length of the beam, effects on the longitudinal strain are very minor; in particular, if the condition of small deformation is satisfied, then warping between adjacent cross-sections does not differ significantly. Therefore, unless shear forces are quite large and change very quickly, the results of pure bending analysis closely match the longitudinal behavior of non-uniform bending. This is very common in long and slender beams; shear deformation is typically only substantial in **stocky beams**, those with large cross-sectional areas relative to their lengths.

4.4.1 Transverse Shear Stress

The presence of shear forces requires shear stresses (τ_{xy} in our coordinate system) acting on the cross-section of the beam. This stress is commonly called the **transverse shear stress**. While the exact distribution of this shear force is not obvious, if the cross-section of the beam is rectangular then the orientation of the force and stress is predictably and consistently parallel to the sides of the beam. We further *assume* that the force is uniformly distributed across the width (z -direction) of the beam, as shown in Figure 4.12. We make no assumptions about the distribution over the depth (y -direction).

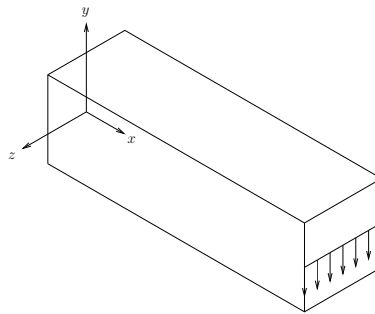


Figure 4.12: Shear force distribution

Consider a beam subjected to non-uniform bending, as in Figure 4.13(a). Non-uniform bending causes the internal moment to vary along the length. A differential element of the beam will experience a different moment, and

therefore a different bending stress, on opposite faces (Figure 4.13(b)). There are also transverse shear stresses acting along the faces of the element, and the applied load on top (not shown for clarity). Keep in mind that at this point, the transverse shear stresses are unknown but the normal bending stresses are known in terms of the moment. We can use this to our advantage. We take

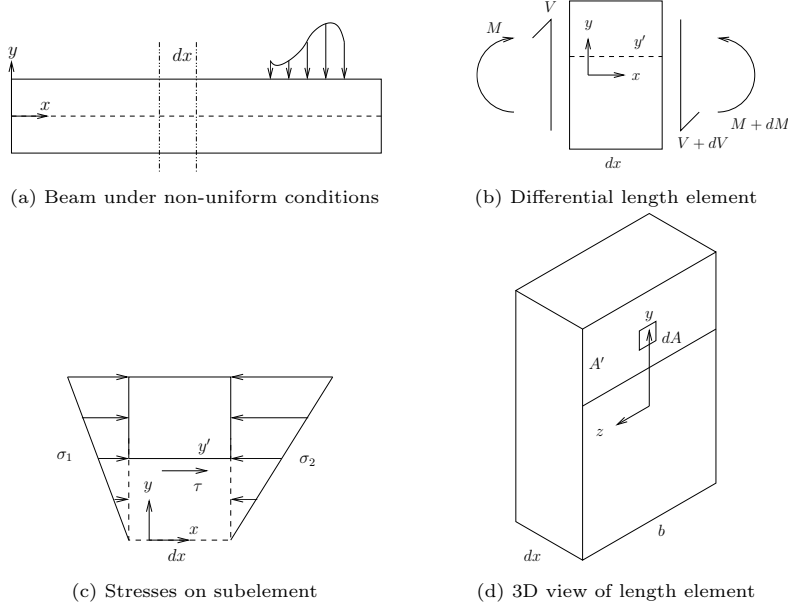


Figure 4.13: Determination of shear stresses in a beam

another section at some point y' and consider the new, smaller element shown in Figure 4.13(c). Note that the figure is drawn from the neutral axis up, showing only the top half of the beam. The sides of this element still experience the normal and shear stresses as the larger element. We can observe that there are no forces acting on the top exposed surface parallel to that surface, so there is no shear stress there. We cannot conclude that for the interior surface at y' , so an unknown shear stress τ is included. The forces and stresses acting in the y -direction are still present, but are irrelevant for further analysis and excluded from the diagram in Figure 4.13(c).

Since the beam is static, force equilibrium must hold. Since we neglect the warping and rotation of the cross-sections and assume small deformation, the normal stresses point in the x -direction. The force on each face of the element can be computed from the stress. For a small area dA , as shown in Figure 4.13(d), the force is:

$$dF = \sigma dA \tag{4.31}$$

The total force on each face is then the sum of the differential forces.

$$F = \int_{A'} \sigma dA \quad (4.32)$$

where A' is the area of the yz -face of the element shown in Figures 4.13(c) and (d). The shear stress on the bottom surface of the element is also oriented in the x -direction. Recall from Section 1.2.1 and Figure 1.21, a shear stress on the negative y -face in the negative x -direction is positive. Thus, the shear stress shown in Figure 4.13(c) is negative. If we *assume it is uniform across the width of the beam*, then the force is given by:

$$F_\tau = \tau b dx \quad (4.33)$$

Taking equilibrium in the x -direction:

$$\begin{aligned} \sum F_x = 0 &= F_1 + F_2 - F_\tau \\ &= \int_{A'} \sigma_1 dA - \int_{A'} \sigma_2 dA - \tau b dx \end{aligned} \quad (4.34)$$

The normal stress σ_1 and σ_2 are defined in terms of the bending moment. Substituting equation (4.19):

$$\int_{A'} \left(-\frac{My}{I} \right) dA - \int_{A'} \left(-\frac{(M + dM)y}{I} \right) dA - \tau b dx = 0 \quad (4.35)$$

I is the second area moment of the entire cross-section, not just A' . Combining the integrals and rearranging:

$$\begin{aligned} \tau b dx &= - \int_{A'} \left(\frac{y dM}{I} \right) dA \\ &= -\frac{dM}{I} \int_{A'} y dA \end{aligned} \quad (4.36)$$

Finally, solving for τ :

$$\tau = -\frac{dM}{dx} \frac{1}{Ib} \int_{A'} y dA \quad (4.37)$$

It will be shown in the next section that dM/dx is the shear force V . The integral is recognizable as the first area moment of the area A' , as detailed in Section 1.1.5.2. With this, the shear stress reduces to:

$$\tau = -\frac{VQ}{Ib} \quad (4.38)$$

This is the shear stress along the bottom surface of A' . We are interested in the transverse shear on the face of the cross-section. Fortunately, due to symmetry of the shear stresses, shear stresses on orthogonal surfaces are equivalent

and equation (4.38) defines the transverse shear stress at any point y' along the cross-section of the beam.

Example 4.3 (Transverse Shear Stress Distribution).

PROBLEM STATEMENT:

Determine the transverse shear stress as a function of y for a rectangular beam cross-section.

KNOWNs:

- Beam width b
- Beam depth h
- Shear force V

ASSUMPTIONs:

- Beam is long and slender
- Beam is deep ($h \gg b$)
- Small deformation
- Neglect warping and rotation of the cross-section

ANALYSIS:

With these assumptions and the rectangular cross-section, this problem satisfies the requirements for *beam analysis*. Consider the transverse shear stress defined by equation (4.38). V , I , and b are all constants with regards to a given cross-section. Only the first area moment Q varies with distance from the neutral axis. Since this is symmetric, the neutral axis is located in precisely the center. To determine Q , choose an arbitrary point y in the cross-section (Figure 4.14 and apply Equation (1.44):

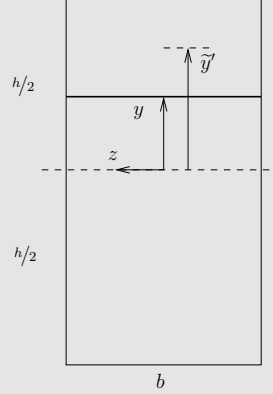


Figure 4.14: Example 4.3

$$Q = A' \tilde{y}' \quad (4.39)$$

The area A' and the centroid of that area \tilde{y}' are given by:

$$A' = \left(\frac{h}{2} - y \right) (b) \quad (4.40)$$

$$\tilde{y}' = y + \frac{h/2 - y}{2}$$

Substituting these into Equation (4.39):

$$Q = b \left(\frac{h}{2} - y \right) \left(\frac{y}{2} + \frac{h}{4} \right) = b \left(\frac{h^2}{8} - \frac{y^2}{2} \right) \quad (4.41)$$

The first area moment, and therefore the transverse shear stress, is quadratic in y . It takes a minimum value of 0 at the top and bottom surfaces of the beam, and reaches a maximum magnitude at the neutral axis.

$$\tau_{max} = -\frac{VQ(y=0)}{Ib} = -\frac{Vb(h^2/8 - 0)}{(bh^3/12)b} = -\frac{3}{2} \frac{V}{hb} \quad (4.42)$$

Two new assumptions were made in deriving this relation that deserve additional attention:

1. The cross-section of the beam is rectangular: In this case, we can comfortably say that the shear force and shear stress are oriented parallel to the y -axis at all points in the cross-section. There are no components of the shear that resolve into a τ_{xz} term and no change in direction at any point. When we take static equilibrium, there is no need to consider a

stress/force on the xy -plane.

2. We assumed the stress is uniform across the width of the beam. This allowed us to reduce the shear force at y' to Equation (4.33). Strictly speaking, this is not true. More detailed analyses of the shear force shows that it reaches peak values at the outer edges ($\pm z$) and a minimum value at the y -axis. If the beam is deep, then there is not much distance between the positive and negative z -faces of the beam, and the variation is quite small. However, if the beam is *flat* then there is a significant z -dependence of the shear stress. Define the width-depth ratio as:

$$\beta = \frac{b}{h} \quad (4.43)$$

For $\beta = 0.5$ (a beam twice as deep as it is wide), the transverse shear stress formula (4.38) has a maximum error of around 3%. However, if $\beta = 2$ (a beam twice as wide as it is deep), that error increases to more than 40%.

4.4.1.1 Sign Convention

One final point of discussion is critical for understanding the transverse shear stress. In Figure 4.13(b), the shear force V on the right face is drawn downward. This produces a negative shear stress τ on that face. The transverse shear stress computed from Equation (4.38) is τ_{xy} . By convention, the sign in equation (4.38) is usually ignored. In practice if it's needed, the orientation of the stress (positive or negative) is generally obvious from direct observation. In most references, the transverse shear stress in a beam will be given as:

$$\tau = \frac{VQ}{Ib} \quad (4.44)$$

with shear force V positive downward on the right face of a material. That being said, for the remainder of this text, the appropriate xy signage will be used for the transverse shear stress. As will become clear in later chapters, the sign of the stress is critical for accurate solution of more complex problems. By including the correct sign at this point, we avoid unnecessary corrections later.

4.4.2 Shear Stress in Non-Rectangular Cross-Sections

The shear formula (4.38) holds for rectangular cross-sections because the stress is parallel to the y -axis at all points, and approximately uniform across the width if β is small enough. If the cross-section is not rectangular, the orientation of the shear stress may change. Consider the circular cross-section shown in Figure 4.15(a). If the shear stress on the face at point A is perfectly vertical, then it can be decomposed into components tangential and normal to the edge. A shear stress that is normal to the edge must be paired with a perpendicular stress due to shear symmetry. This would result in a shear stress on the sides or top of the beam, which we know is not the case.

Instead, Figure 4.15(b) shows that transverse shear stresses are always tangential to the surface of the beam. This means that in general, the shear stress is not parallel to the y -axis in non-rectangular beams. That being said, at certain points it can be. Consider the horizontal diameter (the neutral axis) of the circle in Figure 4.15(b). At this point, tangent lines to the circle are vertical. If we assume that the stress is uniformly distributed across the diameter, then equation (4.38) can be used *at this point*. This is actually convenient, because more detailed analyses show that the peak shear stress actually occurs at this location. Interestingly, although the cross-section seems rather wide here, the error in equation (4.38) is only about 5%. Note that even at the neutral axis, the stress is only vertical, but not truly uniform in the z -direction as shown in Figure 4.15(b). We assume it is for the shear formula (4.38), but in reality that is only the average value of the true stress.

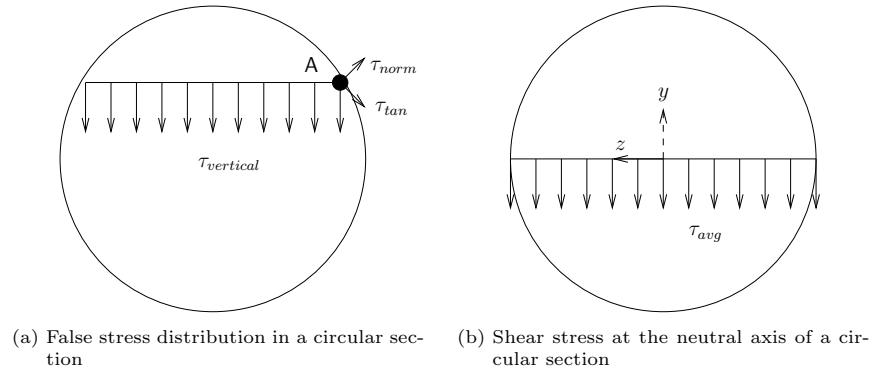


Figure 4.15: Shear stress behavior in non-rectangular sections

In general, equation (4.38) will provide a reasonable (although not exact) value for the shear stress at locations where the stress is parallel to the y -axis. Additional care should be taken if these locations are flat or if the area changes suddenly nearby, as this could lead to a stress concentration.

4.4.3 Shear Stress in Composite Cross-Sections

Many beams are complex shapes constructed of simpler pieces, such as those shown in Figure 4.2. Each of these beams are made of rectangular components with sides parallel to the y -axis, so use of equation (4.38) is appropriate and generally straightforward. However, there are a few points to keep in mind that are easily illustrated using a box beam (Figure 4.16).

A box beam is essentially a hollow rectangular beam. Following Section 4.4.1, the maximum transverse shear stress occurs at the neutral axis. Application of equation (4.38) at the neutral axis is direct.

$$\tau = -\frac{VQ}{Ib} \quad (4.45)$$

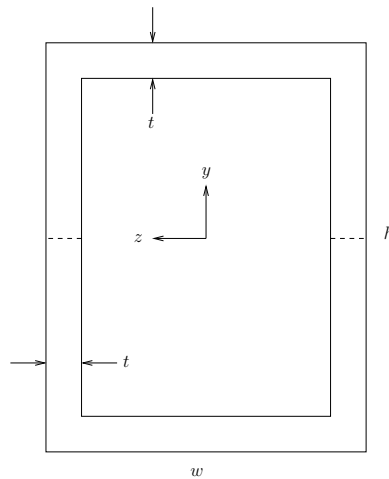


Figure 4.16: Shear stress in a box beam

Q and b require special attention. The first area moment and width dimension are both computed based on the *cross-section geometry*.

$$\begin{aligned} b &= 2t \\ A' &= 2(h/2)t + (w - 2t)(t) \end{aligned} \tag{4.46}$$

Following logically from equation (4.33), b is the width of the *material* at the point where the stress is computed; at the neutral axis, the total width of both webs. The area A' is the area of the material above the neutral axis, not just half the rectangle. Similarly, The centroid of A' must be computed using the composite body formula equation (1.41).

Each of the webs are *deep*, so there is no trouble applying the shear formula in them. However, the flanges of the beam are quite flat. The shear formula would produce substantial error in that region. Furthermore, the stress is concentrated at the web-flange transition, so all simple analyses are limited. However, that is generally not a concern from a practical point of view; in rectangular beams, the maximum transverse shear stress occurs at the neutral axis. In an I-beam, for example, this is far from the flanges. In fact, shear stress in the flanges is very low due to proximity to the top and bottom surfaces. Since the web typically carries the majority of the shear load, it is often more specifically named the **shear web**.

4.5 Internal Shear Force and Bending Moments

In previous sections we have related the normal bending stress and transverse shear stress to the internal moment and shear force, but we have yet to determine those quantities. Internal loadings in beams are rather more complicated than

those in bar and shaft problems due to the 2D nature of the external loads. In the most general case, at any point in a 2D problem there is an unknown force with unknown in-plane orientation, and an unknown moment normal to the plane. The force is typically decomposed into components along given directions. In the case of internal forces, it is generally more useful to consider components tangential and normal to a section, rather than explicit x - and y -components of the force. In this way, we define three internal loads: **internal normal force**, **internal shear force** (tangential component), and **internal bending moment**.

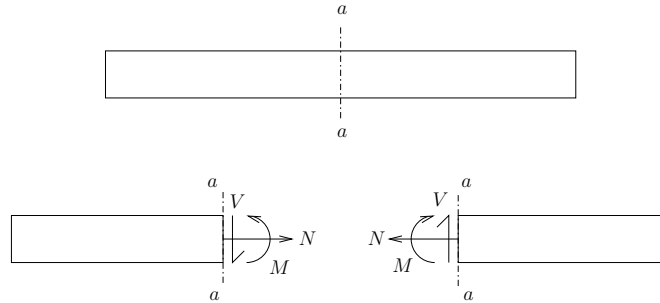


Figure 4.17: Internal forces and moments in a beam

The internal loads are concentrated at the neutral surface. Note that unlike bars and shafts, internal forces and moments are not uniformly distributed over the cross-section in beams. The loads considered here are statically equivalent (i.e., total) forces and moment taken at the neutral surface. Various assumptions and conclusions about the distribution of these forces have been made in Sections 4.1 – 4.4. Consider the beam shown in Figure 4.17. A section $a - a$ is made and the resulting free-body diagrams are shown. According to Newton's third law of motion, the loads on either side of the section must balance, so have the same magnitude with opposite orientation. Orientation and sign conventions are chosen for consistency with deformation:

- Internal normal force is positive *outward* from a face, associated with positive elongation.
- Internal bending moment is positive when the section *bends upward*. Positive moments are related to positive curvature of the beam.
- Internal shear force is positive when it induces a *clockwise* rotation of the section. This is perhaps the least intuitive of the three internal loads, because it is associated with a negative shear stress, and therefore negative shear strain. However, when defined in this way, a positive shear force is related to a positive change in bending moment, as will be shown shortly.
- These sign conventions refer to the whole vector. Cartesian components of the vector follow standard, right-handed coordinate sign rules. For

example, the shear force V is positive downward on the right face of a material section, but the y -component of the force V_y is negative.

Determination of internal shear force and bending moment is procedurally very similar to finding internal normal forces and torques. A section is taken every time external loading changes, and static equilibrium is evaluated. In the case of beams, moments present a slight complication. The magnitude of a moment depends on the distance from forces, therefore moments will change continuously with distance from applied loads. Much like with distributed loads in bars (Example 2.2), we define a distance coordinate to allow continuous variation of the moments. The following examples illustrate the procedure.

Example 4.4 (Internal Loads: Point Force).

PROBLEM STATEMENT:

Determine the internal load field for the simply-supported beam shown in Figure 4.18.

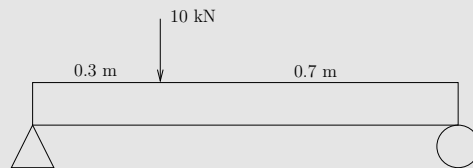


Figure 4.18: Example 4.4

KNOWNs:

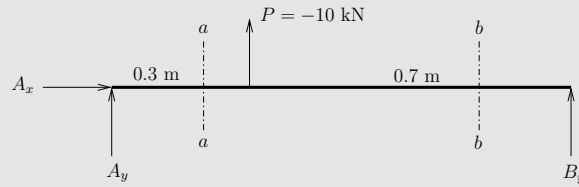
- Beam is simply supported.
 - x - and y -forces at pin support
 - y -force at roller support
- Length of 1 m
- Point load applied 0.3 meters from the left edge

ASSUMPTIONs:

- Beam is in static equilibrium

ANALYSIS:

To determine internal loads, the external reactions must first be found. Consider the free-body diagram shown in Figure 4.19(a). Applying the equations of static equilibrium:



(a) Free-body diagram of full beam

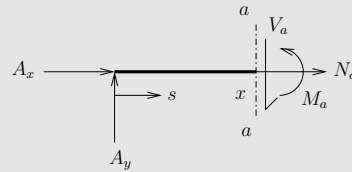
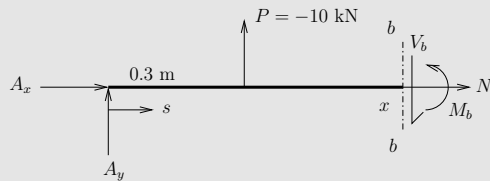
(b) Section $a - a$ (c) Section $b - b$

Figure 4.19: Free-body diagrams

$$\begin{aligned}\sum F_x = 0 &= A_x \\ \sum M_{OA} = 0 &= (-10)(0.3) + (B_y)(1) \implies B_y = 3 \text{ kN} \\ \sum F_y = 0 &= A_y + B_y \implies A_y = 7 \text{ kN}\end{aligned}\quad (4.47)$$

A section must be taken every time the external loading changes, so consider sections $a - a$ and $b - b$ on either side of the applied force and draw free-body diagrams as shown in Figures 4.19(b) and 4.19(c). The sections are taken at variable locations x . Since there is no integration required for this problem, the coordinate s (as in Example 2.2) is unnecessary, although it's included in the figures for completeness.

Section $a - a$:

$$\begin{aligned}\sum F_x = 0 &= N_a \\ \sum F_y = 0 &= A_y + V_{ay} = 7 - V_a \implies V_a = 7 \text{ kN} \\ \sum M_{Oa-a} = 0 &= M_a + (A_y)(-x) \implies M_a = 7x \text{ kN-m}\end{aligned}\quad (4.48)$$

Section $b - b$:

$$\begin{aligned}\sum F_x = 0 &= N_b \\ \sum F_y = 0 &= A_y - 10 + V_{by} = -3 - V_b \implies V_b = -3 \text{ kN} \\ \sum M_{O_{b-b}} = 0 &= M_b + (A_y)(-x) + (-10)(-(x - 0.3)) \\ &\implies M_b = 3 - 3x \text{ kN-m}\end{aligned}\tag{4.49}$$

Thus, the internal shear force and bending moments are fully-defined as piecewise functions of x :

$$V(x) = \begin{cases} 7 \text{ kN} & 0 \leq x < 0.3 \text{ m} \\ -3 \text{ kN} & 0.3 < x \leq 1 \text{ m} \end{cases}\tag{4.50}$$

$$M(x) = \begin{cases} 7x \text{ kN-m} & 0 \leq x < 0.3 \text{ m} \\ 3 - 3x \text{ kN-m} & 0.3 < x \leq 1 \text{ m} \end{cases}\tag{4.51}$$

Example 4.5 (Internal Loads: Distributed Force).

PROBLEM STATEMENT:

Determine the internal load field for the simply-supported beam shown in Figure 4.20.

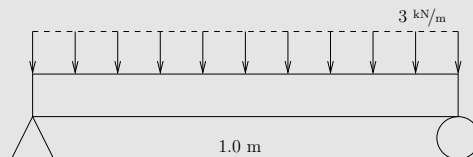


Figure 4.20: Example 4.5

KNOWNs:

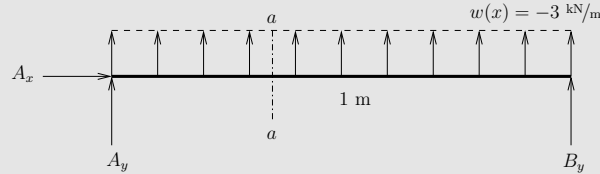
- Beam is simply supported.
 - x - and y -forces at pin support
 - y -force at roller support
- Length of 1 m
- Uniform distributed load $w(x) = -3 \text{ kN/m}$ over entire length

ASSUMPTIONS:

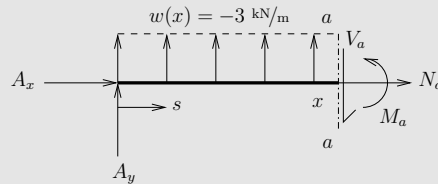
- Beam is in static equilibrium

ANALYSIS:

To determine internal loads, the external reactions must first be found. Consider the free-body diagram shown in Figure 4.21(a). The equations of static equilibrium may be applied here, or by observing symmetry of the load it is clear that reactions at the left and right supports must be equal to half the applied force:



(a) Free-body diagram of full beam



(b) Free-body diagram of full beam

Figure 4.21: Free-body diagrams

$$\sum F_x = 0 = A_x$$

$$A_y = B_y = -\frac{1}{2} \int_0^{1 \text{ m}} -3 \, dx = 1.5 \text{ kN} \quad (4.52)$$

In this example the applied load is consistent over the entire length of the beam, so only a single section is required. Take the section at an arbitrary location x and draw the free-body diagram as shown in Figure 4.19(b) with coordinate s . Consider equilibrium of the section.

$$\sum F_x = 0 = N_a \quad (4.53)$$

$$\begin{aligned} \sum F_y = 0 = A_y + \int_0^x -3 ds + V_{ay} = 1.5 - 3x - V_a \\ \implies V_a = 1.5 - 3x \text{ kN} \end{aligned} \quad (4.54)$$

When computing the moment, the centroid of the distributed force will be required. Since the distributed force is uniform, the centroid is directly in the center.

$$\begin{aligned} \sum M_{O_{a-a}} = 0 = (A_y)(-x) + \left(\int_0^x -3 ds \right) \left(-\frac{x}{2} \right) + M_a \\ = -1.5x + 1.5x^2 + M_a \implies M_a = 1.5x - 1.5x^2 \text{ kN-m} \end{aligned} \quad (4.55)$$

Thus, the internal shear force and bending moments are fully-defined as functions of x :

$$\begin{aligned} V(x) &= 1.5 - 3x \text{ kN} \\ M(x) &= 1.5x - 1.5x^2 \text{ kN-m} \end{aligned} \quad (4.56)$$

4.6 Shear and Moment Diagrams

Using the internal shear force and bending moment, it is possible to determine the transverse shear stress and normal bending stress in a beam at any location along the length. In most circumstances, the maximum value of the stress will be the limiting factor in practical applications, and may be determined using the maximum value of the internal load. To this end, it is often easiest to identify the peak values of shear force and bending moment by plotting the curves against x . These plots of internal loading are known as **shear and moment diagrams**.

As an example, we plot the shear force and moments from Examples 4.4 and 4.5, given in equations (4.50), (4.51), and (4.56).

In looking at these plots, we can make some observations. First, notice that a quadratic moment (Figure 4.22(f)) occurs when the shear force is linear (Figure 4.22(d)), and a linear moment (Figure 4.22(e)) is present with constant shear force (Figure 4.22(c)). It *appears* that the shear force is related to the derivative of the moment.

Furthermore, we can see that the linear shear force shown in Figure 4.22(d) occurs when the applied distributed load is constant (Figure 4.22(b)), and the constant shear force of Figure 4.22(c) is present with zero distributed load (Figure 4.22(a)). If this pattern is to be trusted, then the derivative of the internal shear force is somehow related to the external distributed force.

Two simple examples are far too few to draw conclusions, but it does mo-

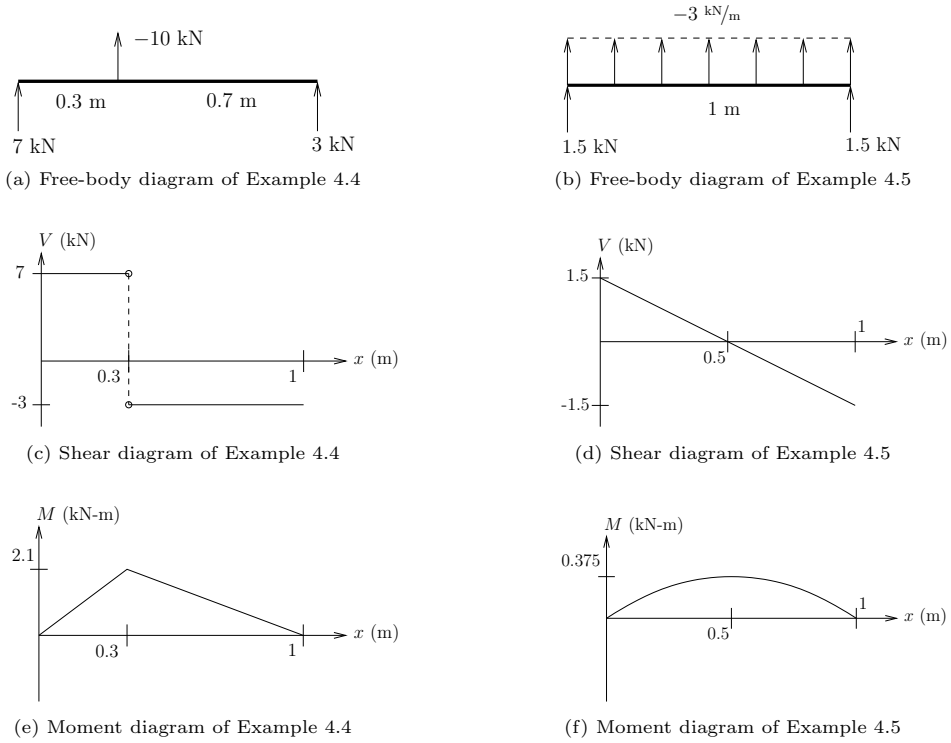


Figure 4.22: Shear and moment diagrams

tivate a more fundamental exploration. Consider a small length of a beam subjected to a distributed load, with sections taken on both sides. As shown in Figure 4.23(a), the internal loading varies across this segment. Because the segment is so small, the distributed load acting on it is approximately uniform (regardless of its macroscopic behavior). Taking static equilibrium:

$$\sum F_y = 0 = V - w dx - (V + dV) \implies \frac{dV}{dx} = w(x) \quad (4.57)$$

$$\begin{aligned} \sum M_{O_a} = 0 &= -M + (w dx) \left(\frac{dx}{2} \right) - (V + dV) dx + M + dM \\ &= \frac{w}{2} dx^2 - V dx - dV dx + dM \end{aligned} \quad (4.58)$$

The differential quantities dx , dV , and dM are much smaller than 1. Their products are therefore negligible relative to the differentials themselves.

$$dx^2, dV dx \approx 0 \quad (4.59)$$

Thus, Equation (4.58) simplifies to:

$$dM - V dx = 0 \implies V(x) = \frac{dM}{dx} \quad (4.60)$$

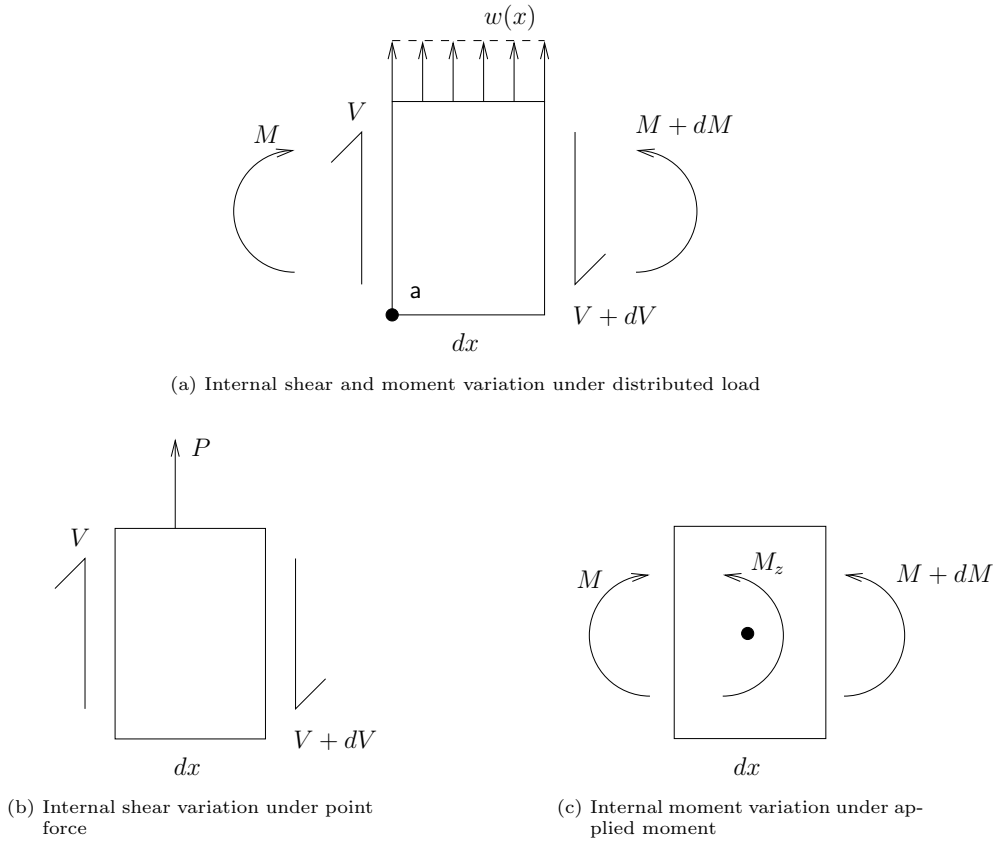


Figure 4.23: Internal load behavior

Equations (4.57) and (4.60) prove the pattern noticed earlier. The magnitude of the distributed load at a point is the slope of the internal shear force at that point. The magnitude of the shear force is in turn the slope of the bending moment at the point. Notice that equation (4.60) validates the statement used in simplifying the transverse shear stress in equation (4.37).

Also necessary to consider are point forces and moments. A point force creates a discontinuity in the internal shear load. To either side of the applied force, the shear force suddenly changes. Likewise, applied moments create discontinuities. This is easily proven following the same procedure as distributed loads. Consider Figures 4.23(b) and (c) with applied loads P and M_z :

$$\begin{aligned} \sum F_y = 0 &= V + P - (V + dV) \implies dV = P \\ \sum M = 0 &= -M + M_z + (M + dM) \implies dM = -M_z \end{aligned} \tag{4.61}$$

The change in internal shear force from left to right due to an applied point load is the magnitude and sign of the force at that point. A point force does

not directly affect the internal moment, but the discontinuity in V leads to a discontinuity in the slope of M . An applied moment also changes the internal moment by its magnitude, but *with opposite sign*. A positive applied moment reduces the internal moment by that amount. As shown in equation (4.61), this is due to the sign convention of our internal shear force and moment.

With this information, it is possible to construct shear and moment diagrams directly from the loading, without taking sections and applying equilibrium.

Example 4.6 (Shear and Moment Diagrams).

PROBLEM STATEMENT:

Draw the shear and moment diagrams for the simply-supported beam shown in Figure 4.18.

KNOWNs:

- Beam is simply supported
 - x - and y -forces at pin support
 - y -force at roller support
- Length of 1 m
- Point load applied 0.3 meters from the left edge

ASSUMPTIONs:

- Beam is in static equilibrium

ANALYSIS:

Begin with a free-body diagram, as shown in Figure 4.24 and solve for external reactions (repeated from Example 4.4):

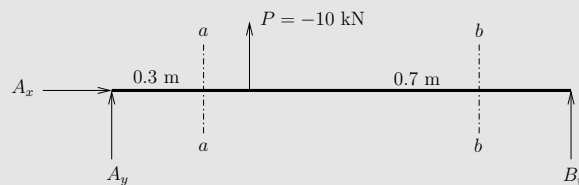


Figure 4.24: Free-body diagram

$$A_y = 7 \text{ kN}; \quad B_y = 3 \text{ kN} \quad (4.62)$$

To draw the shear diagram, begin at the left end of the beam. There is a point force A_y at that location, so the shear force V begins with the value of A_y , 7 kN. There is no distributed load in this portion of the beam, so the slope of the shear diagram is zero (Figure 4.25(a)). At $x = 0.3$ m, there is a point load of -10 kN applied. At this point, the value of the shear force discontinuously decreases by 10 kN to -3 kN. There is still no distributed load, so the slope of the shear curve remains zero to the right end of the beam. The complete shear diagram is shown in Figure 4.25(b).

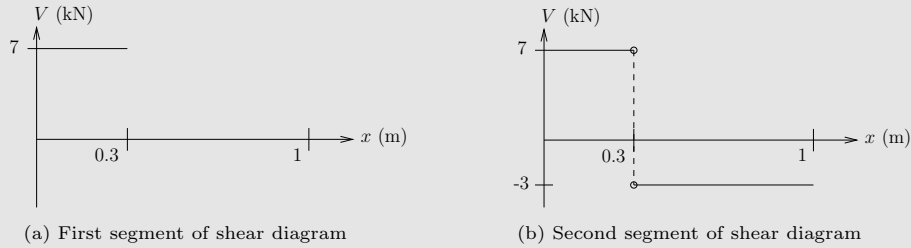


Figure 4.25: Shear force diagram

There is no applied moment at the left end of this beam, so the moment diagram begins at zero. The value of the shear force is a constant in this region, so the slope of the moment will also be constant. Thus, the moment in the left part of the beam is a straight line with slope of 7 (Figure 4.26(a)). This continues until $x = 0.3$ m. Determining the value of the moment at this point is straightforward using the slope:

$$M_{0.3} = M_0 + V(0.3) = 0 + (7)(0.3) = 2.1 \text{ kN-m} \quad (4.63)$$

At this location there is a point force of -10 kN applied. Applied point forces do not directly affect the internal moment, so the value does not change. However, the shear is discontinuous at this point, so the slope of the moment diagram changes accordingly. The slope to the right of this point is -3 . There are no further loads on the beam, so this continues to the right end. Using the slope of this line, the final point is given as:

$$M_1 = M_{0.3} + V(0.7) = 2.1 + (-3)(0.7) = 0 \quad (4.64)$$

Thus, the moment at the right end of the beam returns to zero (Figure 4.26(b)).

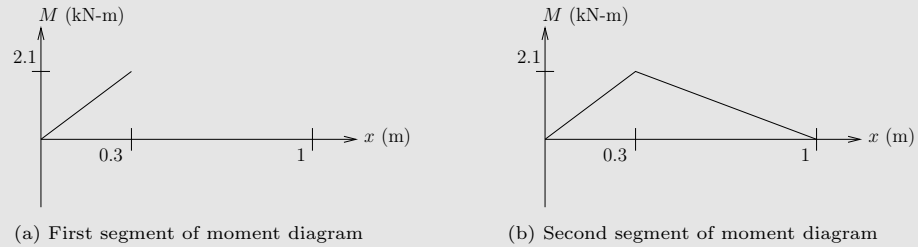


Figure 4.26: Bending moment diagram

The shear and moment diagrams found here match those in Example 4.4.

Example 4.7 (Shear and Moment Diagrams).

PROBLEM STATEMENT:

Determine the internal load field for the simply-supported beam shown in Figure 4.20.

KNOWNs:

- Beam is simply supported
 - x - and y -forces at pin support
 - y -force at roller support
- Length of 1 m
- Uniform distributed load $w(x) = -3 \text{ kN/m}$ over entire length

ASSUMPTIONs:

- Beam is in static equilibrium

ANALYSIS:

Begin with a free-body diagram, as shown in Figure 4.27 and solve for external reactions (repeated from Example 4.5):

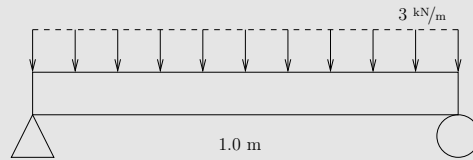


Figure 4.27: Free-body diagram

$$A_y = B_y = 1.5 \text{ kN} \quad (4.65)$$

To draw the shear diagram, begin at the left end of the beam. There is a point force A_y at that location, so the shear force V begins with the value of A_y , 1.5 kN. The distributed load is uniform, so the slope of the shear diagram is a constant (Figure 4.28). There are no other loads applied, so the shear diagram is a single line. Using the slope, the shear at the right end of the beam is easily determined:

$$V_1 = V_0 + w(1) = 1.5 - 3 = -1.5 \text{ kN} \quad (4.66)$$

We observe that the shear force equals zero at $x = 0.5$ m.

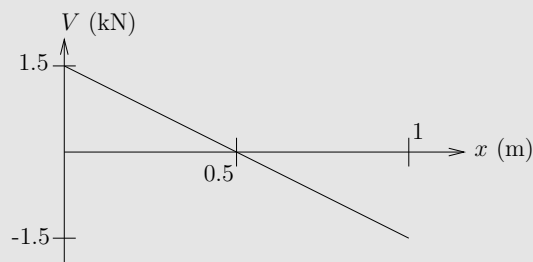


Figure 4.28: Shear force diagram

As before, there is no applied moment at the left end of the beam, so the moment diagram begins at zero. The value of the shear force is continuously changing, so the slope of the moment is continuously changing. Shear force has its greatest positive value at the left end, a zero value in the center, and the greatest negative value at the right end. Applying these values as slopes, the moment steeply increases at the left edge, levels out in the center, and steeply decreases at the right edge. Positive and negative slopes have the same values, so the moment ends at the same value it started, zero (Figure 4.29).

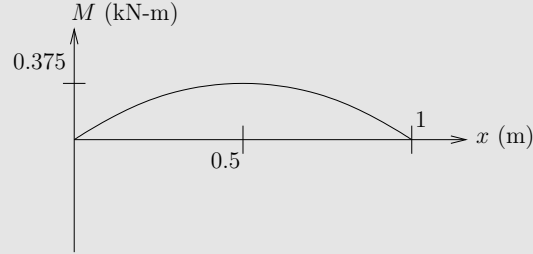


Figure 4.29: Bending moment diagram

Determination of points of interest in this case is rather more complicated. The moment is not a straight line, so a simple slope form is not possible. There are several options though. First, from Equation (4.60) the shear force is the derivative of the moment. Since shear force is linear, the moment must be quadratic. Using basic geometry and algebra, it is possible to define an expression for the parabola. Alternatively, integrating the shear force would yield the moment (just as integrating the distributed load would produce the shear force). This may be simple in some cases, but with complicated loading, especially with various point loads, the integrals may be messy.

Perhaps the most straightforward way to determine the value of the moment at a single point is to take a section precisely at that point. If the location of the section is well-defined, no variable position coordinate is necessary. In this case, we recognize that when the shear force is zero, the moment reaches a stationary point (maximum or minimum). This occurs at $x = 0.5$, so we take a section precisely at that point and solve for the moment at that location. Using Figure 4.21(b) with $x = 0.5$ m:

$$\begin{aligned} \sum M_{O_{a-a}} = 0 &= (A_y)(-0.5) + \left(\int_0^{0.5} -3 ds \right) \left(-\frac{0.5}{2} \right) + M_a \\ &= -0.75 + (-1.5)(-0.25) + M_a \quad \implies \quad M_a = 0.375 \text{ kN-m} \end{aligned} \tag{4.67}$$

This value is shown in Figure 4.29. These shear and moment diagrams match those from Example 4.5.

Drawing shear and moment diagrams by observation is generally straightforward and quick compared to determining the entire shear and moment field from equilibrium of sections. However, if detailed expressions for the shear force and moment are required as functions of x , this approach may be less useful. For complex distributions, it is often quite difficult to extract functional representations from graphs. That being said, in many cases our primary interest may be in determining the maximum stress so as to prevent failure. In such

cases (if the beam is prismatic), the maximum bending moment and maximum shear force directly yield the maximum stresses. Using the direct approach to constructing shear and moment diagrams, we can quickly find those loads and their locations. This is the primary advantage of drawing the diagrams.

To summarize, we use the following rules to construct shear and moment diagrams:

- Point forces create discontinuous changes in the internal shear force at the point of application. The change in shear force has the magnitude and direction of the applied force.
- Applied moments create discontinuous changes in the internal moment at the point of application. The change in internal moment has the magnitude of the applied moment, but opposite sign.
- The shear force takes the value of point loads at the ends of the beam:
 - If there is no point force at the end of a beam, the internal shear goes to zero at that end.
 - The internal shear force at the left end of a beam matches the point force applied there (i.e., the point force discontinuously adjusts the value of the shear from zero).
 - The internal shear force at the right end of a beam matches the *negative* of the point force applied at that end (i.e., the point force discontinuously adjusts the value of the shear back to zero).
- The bending moment takes the *negative* value of point loads at the ends of the beam:
 - If there is no applied moment at the end of a beam, the internal moment goes to zero at that end.
 - The internal moment at the left end of a beam matches the *negative* of the applied moment there (i.e., the applied moment discontinuously adjusts the value of the internal moment from zero).
 - The internal moment at the right end of a beam matches the applied moment at that end (i.e., the applied moment discontinuously adjusts the value of the internal moment back to zero).
- The slope of the shear force curve at any point is the value of the applied distributed load at that point.
- The slope of the moment curve at any point is the value of the shear force at that point.

Example 4.8 (Shear and Moment Diagrams).

PROBLEM STATEMENT:

Draw the shear and moment diagrams directly for the cantilevered beam shown in Figure 4.30.

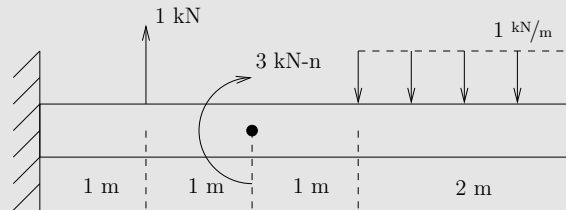


Figure 4.30: Example 4.8

KNOWNs:

- Beam is cantilevered
 - x - and y -forces at support
 - Moment at support
- Length of 5 m
- Point force $P = 1$ kN 1 m from support
- Applied moment $M = -3$ kN-m 2 m from support
- Uniform distributed load $w = -1$ kN/m over final two meters of beam

ASSUMPTIONs:

- Beam is in static equilibrium

ANALYSIS:

Begin with a free-body diagram, as shown in Figure 4.31 and solve for external reactions.

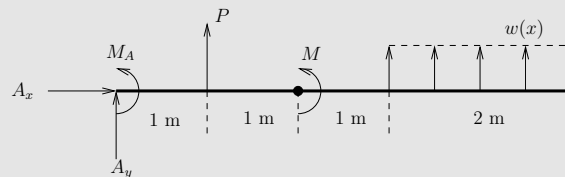


Figure 4.31: Free-body diagram

$$\begin{aligned}
 \sum F_x = 0 &= A_x \\
 \sum F_y = 0 &= A_y + 1 + (-1)(2) \implies A_y = 1 \text{ kN} \\
 \sum M_{OA} = 0 &= M_A + (1)(1) - 3 + (-1)(2) \left(3 + \frac{2}{2}\right) \\
 &\implies M_A = 10 \text{ kN-m}
 \end{aligned}
 \tag{4.68}$$

There is a point load at the left end of the beam, so the shear diagram begins with the value of A_y , 1 kN. There is no distributed load in the first portion of the beam, so the slope is zero and the shear force is constant up to the first external load, illustrated in Figure 4.32(a). At $x = 1$ m, there is a point force with value of positive 1, so the shear discontinuously increases by one at that point. There is still no distributed force, so the slope remains zero to the next applied load (Figure 4.32(b)). At $x = 2$ m, there is an applied moment. Couple moments do not affect the internal shear, and there is still no distributed load, so the internal shear force holds the same constant value through this segment of the beam (Figure 4.32(c)). At $x = 3$ m, a uniform distributed load of -1 kN/m begins. This distributed load continues to the end of the beam, so the shear has a constant slope of -1 out to the end of the beam. There is no point load at the right end, so the shear force returns to zero when $x = 5$ m (Figure 4.32(d)).

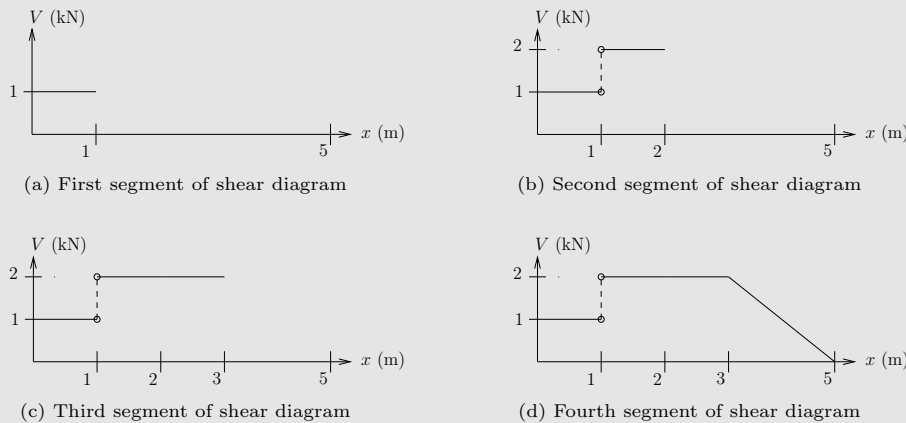


Figure 4.32: Construction of internal shear force diagram

To construct the moment diagram, also begin at the left end, $x = 0$. There is an applied moment here, $M_A = 10$ kN-m. The internal moment therefore begins with a value of -10 kN-m. The shear force has a constant value of 1 kN in the first segment, so the slope of the internal moment in this segment is 1 (Figure 4.33(a)). Using the slope of this line, we can easily determine the value

where this segment ends:

$$M_1 = M_0 + V\Delta x = -10 + (1)(1) = -9 \text{ kN-m} \quad (4.69)$$

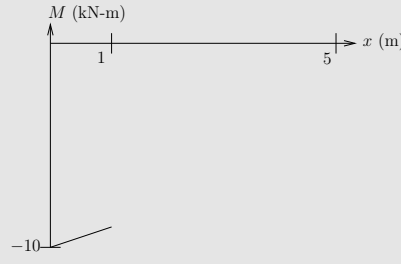
At $x = 1$ m, a point force is applied. This has no direct effect on the internal moment, but the value of the shear force changes here. Thus, the slope of the moment diagram suddenly increases to 2 at this location, and remains constant through the segment (Figure 4.33(b)).

$$M_2 = M_1 + V\Delta x = -9 + (2)(1) = -7 \text{ kN-m} \quad (4.70)$$

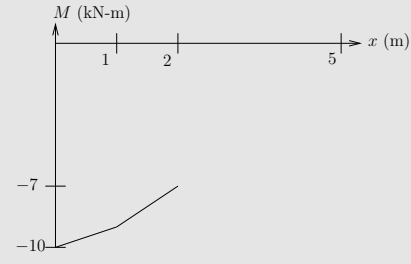
At $x = 2$ m, there is a point moment $M = -3$ kN-m applied. This point moment creates a jump in the value of the internal moment with the opposite sign, so the internal moment increases by 3. The shear force is unchanged, so the slope of the moment remains the same (Figure 4.33(c)).

$$M_3 = M_2 - M + V\Delta x = -7 - (-3) + (2)(1) = -2 \text{ kN-m} \quad (4.71)$$

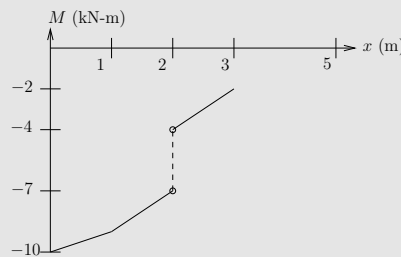
In the final segment of the beam, the shear force decreases linearly. Therefore the slope of the moment is relatively steeply positive at $x = 3$ m, and decreases smoothly to 0 at the right end of the beam. Since there is no applied moment at this point, the value of the internal moment also reaches 0 at $x = 5$ m.



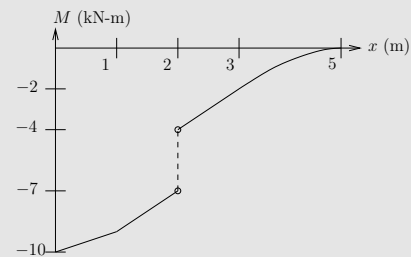
(a) First segment of moment diagram



(b) Second segment of moment diagram



(c) Third segment of moment diagram



(d) Fourth segment of moment diagram

Figure 4.33: Construction of internal moment diagram

As one final point of discussion, pay attention to the discontinuities present when point loads are applied. As discussed in Section 1.3.1, point loads themselves are non-physical. They are simply models representing physical distributed loads. If we consider a physical load, we can introduce continuity into the internal shear force and moments. First consider a single point load, as shown in Figure 4.34(a). A likely physical distribution equivalent to this point load is shown in Figure 4.34(b). The total magnitude of this distributed force is the same as the point load. The shear diagrams for each load are shown in Figures 4.34(c) and (d). In the case of the point load, the internal shear suddenly changes value at the point of application. For the distributed load, the internal shear force smoothly transitions through the same value (because the total applied force is the same). When the distributed force acts over a small area, the discontinuous jump of Figure 4.34(c) may be a reasonable approximation. Furthermore, when we look at either side of the distributed point load, the internal shear force is the same as if we look at either side of the point load.

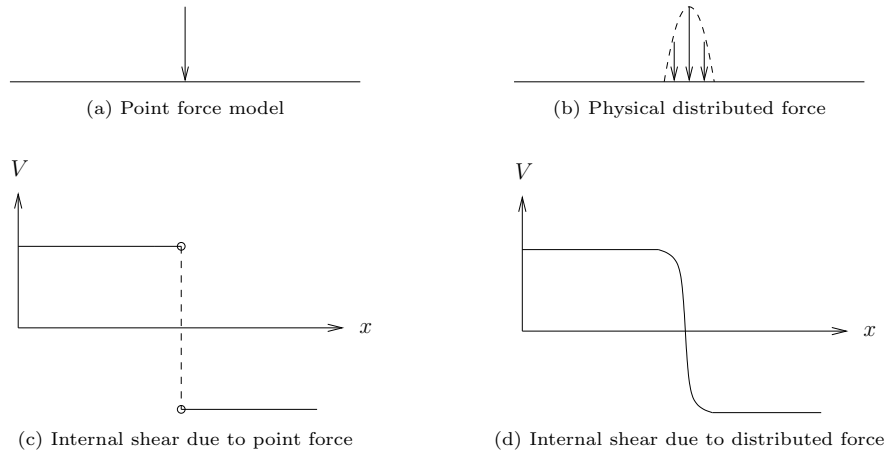


Figure 4.34: Physical effects of point forces

Similarly, we can consider an applied moment as in Figure 4.35(a). Physically, a moment is produced by couple forces that balance, such as those in Figure 4.35(b). The applied moment has no effect on the shear force, so the shear diagram, Figure 4.35(c), does not show the moment at that point. However, the couple forces are distributed loads, and so do result in internal shear force (4.35(d)). This shear force increases along the first part of the couple, but decreases at the same rate when it reaches the other part of the couple because it is oriented in the opposite direction. Thus the shear force returns to zero after the couple. In considering the internal moment, the discontinuity is immediately apparent in the first case shown in Figure 4.35(e). Following the shear diagram in Figure 4.35(d), when a physical couple is applied Figure 4.35(f) shows a smooth transition over the magnitude of the couple. Outside the range of the couple, the two moment diagrams show exactly the same behavior.

Thus, in the case of internal moments, an applied moment can be a convenient approximation of couple forces, especially those acting over a small area.

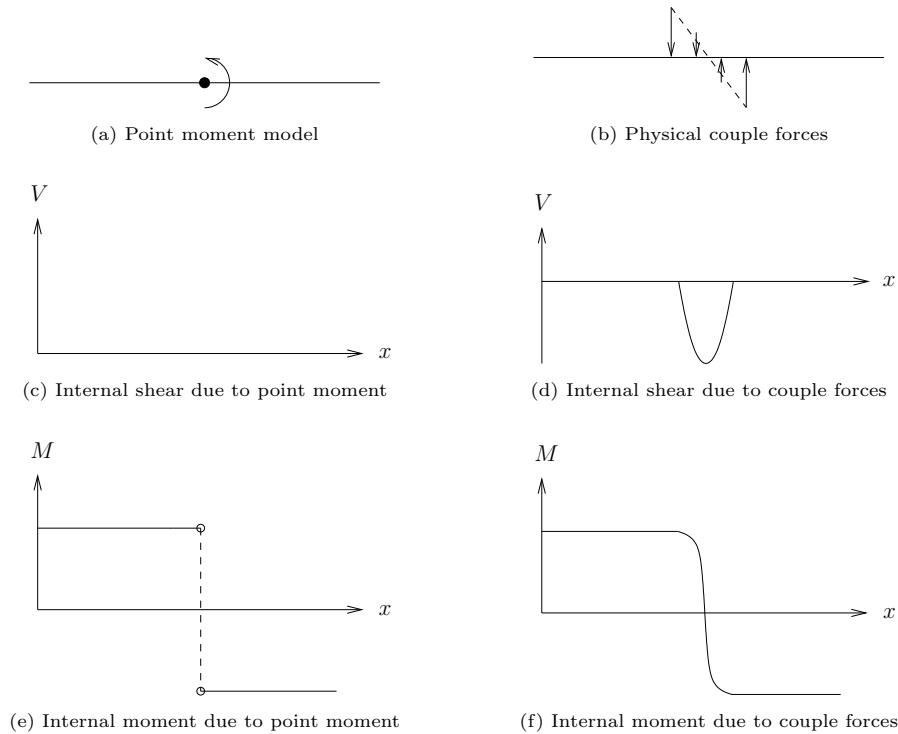


Figure 4.35: Physical effects of point moments

4.7 Euler-Bernoulli Beam Theory

Beams are structural members subjected to transverse loading. As such, they deform by bending (Figure 4.36(a), greatly exaggerated to show detail). One way to measure the bending deformation is the curvature of the beam. The curvature is directly related to both the internal moment and the normal stress, and is easily extracted from equations (4.8) or (4.16). However, curvature is often difficult to measure and may not be a useful measure of deformation in practice. When considering a loaded beam, it is generally much easier to identify and measure its **deflection**, or vertical displacement, $v(x)$. That being said, the two measures should intuitively be related.

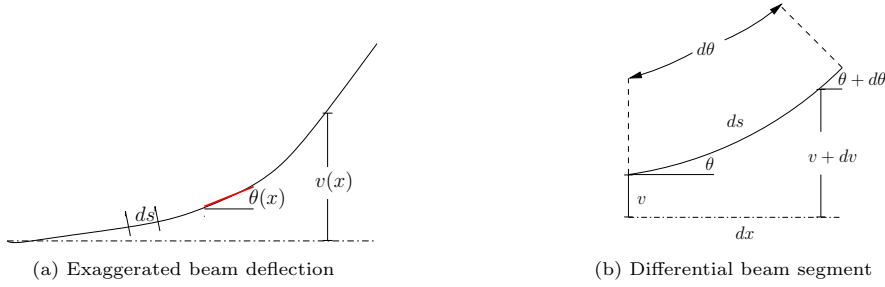


Figure 4.36: Beam deformation

If the deflection of a beam varies from left to right, the slope of the beam (or neutral surface) will vary as well. At a point, the slope is defined as the first derivative of the deflection.

$$\text{slope} = \frac{dv}{dx} \quad (4.72)$$

where dv is the change in deflection of the neutral surface over a differential length dx . More specifically, this is the slope of a tangent line to the deflection curve at this point, also shown in Figure 4.36(a). The slope is the tangent of the angle the tangent line makes with the x -axis.

$$\tan(\theta) = \frac{dv}{dx} \quad (4.73)$$

Since we assume the beam only undergoes small deformation, the angle θ will be small. For small angles, the tangent of the angle is approximately the angle itself.

$$\theta \approx \frac{dv}{dx} \quad (4.74)$$

The angle of orientation of the neutral surface (in radians) is equivalent to the first derivative of the deflection.

Consider a differential segment of the neutral surface, shown in Figure 4.36(b). As described previously, the deflection and orientation will vary between the endpoints of the segment. Recall that for this differential length, the radius of curvature is a constant; in other words, the segment is a circular arc with radius ρ . Thus, if two rays are drawn normal to the neutral surface at two points on the differential segment, the angle between them will be the same as the change in orientation of the neutral surface between those two points, $d\theta$. Logically, orientation can be related to curvature! From equation (4.3):

$$\kappa = \frac{d\theta}{ds} \quad (4.75)$$

For small deformation, $ds \approx dx$, so curvature becomes:

$$\kappa = \frac{d\theta}{dx} \quad (4.76)$$

Substituting Equation (4.74) into this relation, the curvature is related to the deflection.

$$\kappa = \frac{d^2v}{dx^2} \quad (4.77)$$

Our various measures of deformation are now related to each other, but ideally we would like to relate them to the applied loading on the beam. From equation (4.16):

$$\kappa = \frac{M}{EI} \quad (4.78)$$

Or:

$$M = EI \frac{d^2v}{dx^2} \quad (4.79)$$

The internal moment M is related to the second derivative of the vertical deflection. By solving this ordinary differential equation, deformation of the beam is extracted from the loads that produce the internal moment. Note that while we have explicitly only used the assumption of small deformation here, several others are still necessary. Particularly, in using the moment-curvature relationship (4.16), all of the assumptions required to derive it must continue to hold here. Most important (and perhaps most problematic) is the assumption that cross-sections remain planar and normal to the neutral surface throughout deformation.

It is possible to continue development and relate the deflection directly to applied loads. From equation (4.60), we know that the shear force is related to the derivative of the moment.

$$V = \frac{dM}{dx} = \frac{d}{dx} \left[EI \frac{d^2v}{dx^2} \right] \quad (4.80)$$

Furthermore, the applied distributed load is defined in equation (4.57) as the derivative of the internal shear force.

$$w = \frac{dV}{dx} = \frac{d^2}{dx^2} \left[EI \frac{d^2v}{dx^2} \right] \quad (4.81)$$

Equation (4.81) is known as the **Euler-Bernoulli beam theory** for a static beam. It is a fourth-order ordinary differential equation relating applied distributed force to deflection of a beam.

4.7.1 Application of Euler-Bernoulli Beam Theory

If the beam is homogeneous and prismatic, the **flexural rigidity** EI is constant and can be removed from the derivative. In this case, the differential equation is separable and can be solved by direct integration. When starting from equation (4.81), integration can be complicated by the presence of point loads. In practice, it is typically more straightforward to use the applied loads to determine the internal moment, and integrate equation (4.79) directly. Regardless of the starting point, if solved correctly the deflection curve will be the same in either case.

Beginning with the moment deflection equation (4.79), integrate once to find the slope of the beam. If E and I are constant with respect to x , they can be factored out of the integral.

$$\theta(x) = \frac{dv}{dx} = \frac{1}{EI} \int M(x) dx = A(x) + C_1 \quad (4.82)$$

Here, $A(x)$ is the function resulting from integrating the moment, and C_1 is the unknown constant of integration. Integrating the slope equation with respect to x will yield the deflection curve.

$$v(x) = \int \theta dx = \frac{1}{EI} \int (A(x) + C_1) dx = B(x) + C_1x + C_2 \quad (4.83)$$

As before, $B(x)$ is the function resulting from indefinite integration of the slope, and C_2 is the new constant of integration. To completely define the deflection curve, the constants of integration must be evaluated.

Often the moment may be a piecewise function of x , such as:

$$M(x) = \begin{cases} M_1(x) & x < a \\ M_2(x) & x > a \end{cases} \quad (4.84)$$

If we integrate this to get the slope, we end up with two expressions.

$$\theta(x) = \int M(x) dx = \begin{cases} \int M_1 dx & x < a \\ \int M_2 dx & x > a \end{cases} = \begin{cases} A_1(x) + C_1 & x < a \\ A_2(x) + C_2 & x > a \end{cases} \quad (4.85)$$

Integrating the slope to get deflection:

$$v(x) = \int \theta(x) dx = \begin{cases} \int \theta_1 dx & x < a \\ \int \theta_2 dx & x > a \end{cases} = \begin{cases} B_1(x) + C_1x + C_3 & x < a \\ B_2(x) + C_2x + C_4 & x > a \end{cases} \quad (4.86)$$

In this case, there are four constants of integration to determine. Naturally, each segment in a piecewise function will require two additional constants of integration.

4.7.2 Boundary and Continuity Conditions

Constants of integration are evaluated by considering the **boundary conditions**. Boundary conditions are known function values at specific points. In the case of beam deformation, we may know specific values of the deformation at certain points without solving the problem. In other words, we use the constraints provided by the supports to determine the deformation at specific points, much like defining compatibility conditions. For example, the deflection of the beam at a pin support is zero, because the pin holds the beam in place there. However, the beam is free to rotate about the pin, so there is no restriction on the slope. A cantilever support restricts both displacement and rotation, so the deflection and slope of a beam are both zero at that location.

A statically determinate beam has three support reactions. One of these is an x -force that doesn't have any effect on bending deformation. Therefore, there are only two support reactions associated with bending, i.e., there are only two constraints on the deformation. The boundary conditions can be used to solve for up to two constants of integration. This is fine if the moment is not piecewise-defined, as in equation (4.83). If the moment is defined in the piecewise sense, however, there will be more than two constants of integration. The two boundary conditions of a statically determinate beam are insufficient to solve this problem.

We resolve this issue similarly to defining compatibility conditions in bars and shafts. By carefully observing the problem, we can identify additional conditions that the deformation field must satisfy. It may not be possible to define exact values of deflection or slope, but perhaps we can recognize some general behavior that will bring the problem closure. Consider the deflection curves shown in Figure 4.37(a) – (c). These curves show three different scenarios. Figure 4.37(a) shows the deflection of a beam as smooth and continuous. This is what we would expect in a physical system.

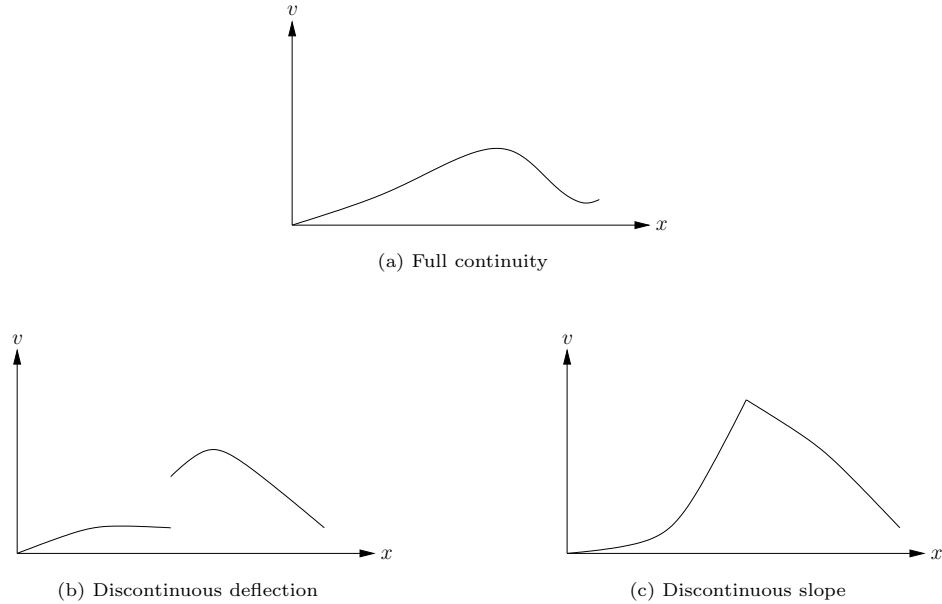


Figure 4.37: Beam deflection scenarios

Figure 4.37(b) shows a discontinuous deflection curve. While this may be admissible mathematically, it presents a fundamental problem for beam analysis. If two adjacent points experience discontinuous deflection, then the beam has broken. One of our most basic assumptions is that we are dealing with continua. Therefore, this scenario must be avoided. In Figure 4.37(c), this problem has been avoided. However, there is now a sharp cusp where the slope of the

deflection changes suddenly. In other words, θ is discontinuous at that point. Physically, there is always some curvature between points. The slope will not change suddenly without beam failure and very high local strains. This is another situation to be avoided.

Fortunately, avoiding these issues gives us the information necessary to manage the additional constants of integration. At every boundary between segments of piecewise-defined deflection and slope curves, we enforce physical behavior through the use of **continuity conditions**:

$$\begin{aligned}\theta_1(a) &= \theta_2(a) \\ v_1(a) &= v_2(a)\end{aligned}\tag{4.87}$$

That is, at the point where the two segments meet, the deflections and slopes must have the same value. In implementing these conditions, we enforce continuity of the slope and deflection curves. We also get two additional equations for every point where the piecewise function segments meet. Since we also get two additional constants of integration at each such location, this brings the mathematical model closure.

Example 4.9 (Beam Deflection).

PROBLEM STATEMENT:

Determine the deflection curve of the beam in Example 4.4, shown in Figure 4.38. The beam is prismatic.

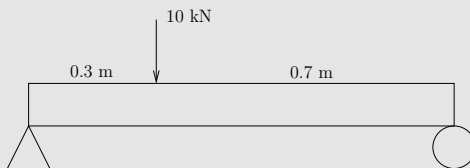


Figure 4.38: Example 4.9 (repeated from Example 4.4)

KNOWNs:

- Beam is simply supported
 - x - and y -forces at pin support
 - y -force at roller support
- Length of 1 m
- Point load applied 0.3 meters from the left edge
- Beam is prismatic

ASSUMPTIONS:

- Beam is in static equilibrium
- Cross-section is deep
- Beam is long and slender
- Material is homogeneous and linearly elastic
- Neglect shear strain
- Small deformation

ANALYSIS:

In order to complete a *beam analysis* of this member, several assumptions were necessary. Without data, we simply assume that the cross-section is deep and shear strains are negligible. With this, the cross-sections of the beam are approximately planar and normal to the neutral axis. If the beam is homogeneous and linearly elastic, the Young's modulus is a constant.

To determine the deflection curve, we first find the internal moment of the beam, as a function of x . This has already been found in Example 4.4, but is repeated here:

$$M(x) = \begin{cases} 7x \text{ kN-m} & 0 \leq x \leq 0.3 \text{ m} \\ 3 - 3x \text{ kN-m} & 0.3 \leq x \leq 1 \text{ m} \end{cases} \quad (4.88)$$

This is a piecewise-defined function, so each portion must be integrated individually.

$$\theta(x) = \frac{1}{EI} \begin{cases} \int M_1(x) dx \\ \int M_2(x) dx \end{cases} = \frac{1}{EI} \begin{cases} \frac{7x^2}{2} + C_1 & 0 \leq x \leq 0.3 \text{ m} \\ 3x - \frac{3x^2}{2} + C_2 & 0.3 \leq x \leq 1 \text{ m} \end{cases} \quad (4.89)$$

To find the deflection, solve Equation (4.74). Integrating the slope:

$$v(x) = \begin{cases} \int \theta_1(x) dx \\ \int \theta_2(x) dx \end{cases} = \frac{1}{EI} \begin{cases} \frac{7x^3}{6} + C_1x + C_3 & 0 \leq x \leq 0.3 \text{ m} \\ \frac{3x^2}{2} - \frac{x^3}{2} + C_2x + C_4 & 0.3 \leq x \leq 1 \text{ m} \end{cases} \quad (4.90)$$

In Equation (4.90) there are four constants of integration to be determined. From Figure 4.38, we see that the beam has a pin at the left end and roller support at the right end. Each of these supports provide vertical force reactions, but no moments. Therefore, they only resist translational motion. The boundary conditions provided by the supports are:

$$\begin{aligned} v(0) &= 0 \\ v(1) &= 0 \end{aligned} \quad (4.91)$$

The slope and deflection are both piecewise-defined with a single interface in curve definitions. This yields the following continuity conditions.

$$\begin{aligned}\theta_1(0.3) &= \theta_2(0.3) \\ v_1(0.3) &= v_2(0.3)\end{aligned}\quad (4.92)$$

Combine equations (4.89) – (4.92).

$$v(0) = v_1(0) = \frac{1}{EI} \left(\frac{7(0)^3}{6} + C_1(0) + C_3 \right) = 0 \quad (4.93)$$

$$v(1) = v_2(1) = \frac{1}{EI} \left(\frac{3(1)^2}{2} - \frac{(1)^3}{2} + C_2(1) + C_4 \right) = 0 \quad (4.94)$$

$$\theta_1(0.3) = \theta_2(0.3) \implies$$

$$\frac{1}{EI} \left(\frac{7(0.3)^2}{2} + C_1 \right) = \frac{1}{EI} \left(3(0.3) - \frac{3(0.3)^2}{2} + C_2 \right) \quad (4.95)$$

$$v_1(0.3) = v_2(0.3) \implies$$

$$\frac{1}{EI} \left(\frac{7(0.3)^3}{6} + C_1(0.3) + C_3 \right) = \frac{1}{EI} \left(\frac{3(0.3)^2}{2} - \frac{(0.3)^3}{2} + C_2(0.3) + C_4 \right) \quad (4.96)$$

Simplifying and gathering equations (4.93) – (4.96):

$$\begin{aligned}C_3 &= 0 \\ C_2 + C_4 &= -1 \\ C_1 - C_2 &= 0.45 \\ 0.3C_1 - 0.3C_2 + C_3 - C_4 &= 0.09\end{aligned}\quad (4.97)$$

Solving this system of equations yields the constants of integration.

$$\mathbf{C} = \begin{Bmatrix} -0.595 \\ -1.045 \\ 0.000 \\ 0.045 \end{Bmatrix} \quad (4.98)$$

Substituting these constants into equation (4.90), the deflection of the beam is fully-defined at every point. The deflection curve is graphed in Figure 4.39.

$$v(x) = \frac{1}{EI} \begin{cases} \frac{7x^3}{6} - 0.595x & 0 \leq x \leq 0.3 \text{ m} \\ \frac{3x^2}{2} - \frac{x^3}{2} - 1.045x + 0.045 & 0.3 \leq x \leq 1 \text{ m} \end{cases} \quad (4.99)$$

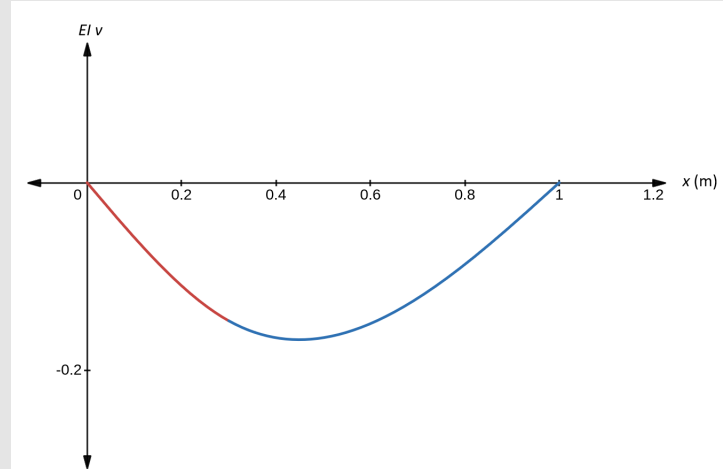


Figure 4.39: Deflection curve

The deflection curve in Figure 4.39 is smooth and continuous. At either end the deflection is zero, showing that boundary conditions are satisfied. Since E and I are not given values in equation (4.99), the deflection shown in the graph is scaled by their product. As a final check, $x = 0.3$ may be substituted into both pieces of equation (4.99) to determine that they are in fact equal at that point.

Beam deflection in many cases can be fairly predictable, but care should be taken when relying solely on personal intuition. In the previous example, one might expect maximum deflection to occur at the location of the greatest moment, $x = 0.3$ m. However, the solution of the Euler-Bernoulli beam equation shows that it is further to the right of that point. It makes sense that this location, further from the restricting effects of the supports, would deflect further but that point can be easy to miss for those in a hurry. Do not jump to conclusions.

If the moment distribution is more complex, as in the following example, it becomes even more challenging to make any detailed predictions.

Example 4.10 (Beam Deflection).

PROBLEM STATEMENT:

Determine the deflection curve of the beam in Example 4.8, shown in Figure 4.40. The beam is steel with a rectangular cross-section 0.5 m deep and 0.1 m wide.

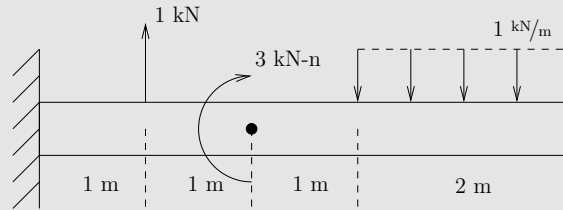


Figure 4.40: Example 4.10 (repeated from Example 4.8)

KNOWNs:

- Beam is cantilevered
 - x - and y -forces at support
 - Moment at support
- Length of 5 m
- Point force $P = 1$ kN, 1 m from support
- Applied moment $M = -3$ kN-m, 2 m from support
- Uniform distributed load $w = -1$ kN/m over final two meters of beam
- Beam is rectangular and prismatic
 - $h = 0.5$ m
 - $b = 0.1$ m
- Steel: $E = 200$ GPa (from Appendix B)

ASSUMPTIONs:

- Beam is in static equilibrium
- Neglect shear strain
- Small deformation

ANALYSIS:

With the dimensions given for this beam, it is long, slender, and deep. With the additional assumptions made, this satisfies the requirements to use *Euler-Bernoulli beam theory*. To apply Euler-Bernoulli theory, the internal moment field must be known. The support reactions were found in Example 4.8. Using

these, the free-body diagram is shown in Figure 4.41(a). Sections are taken every time the external loading changes, as shown in Figures 4.41(b) – (e), and equilibrium applied at each. Lengths are excluded from Figure 4.41 to avoid clutter, so keep in mind that each load is separated by a distance of one meter.

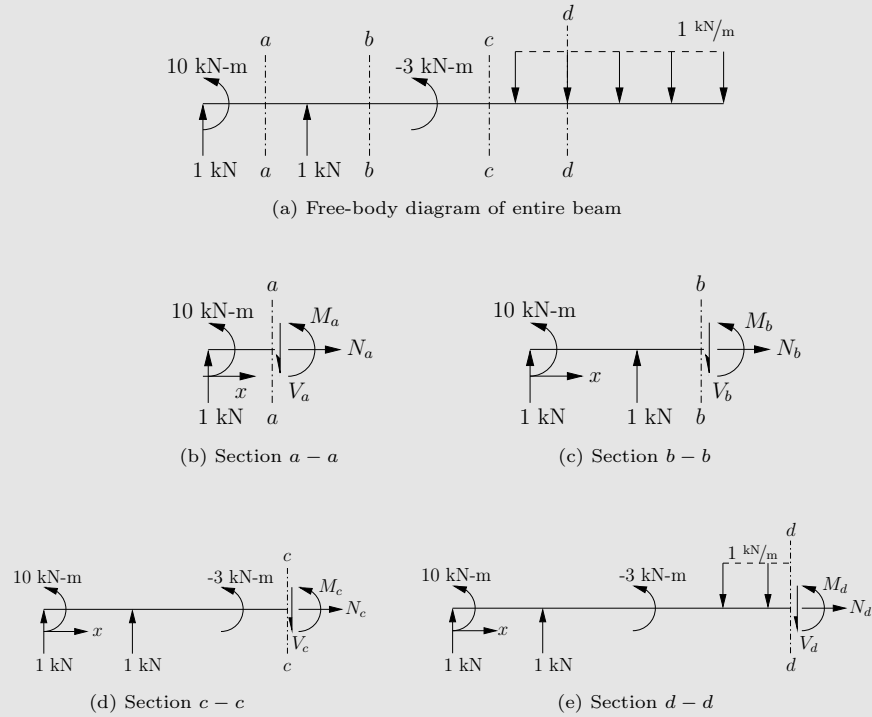


Figure 4.41: Free-body diagrams

Section $a - a$:

$$\begin{aligned}\sum F_x &= 0 = N_a \\ \sum F_y &= 0 = 1 - V_a \implies V_a = 1 \text{ kN} \\ \sum M_{O_{a-a}} &= 0 = 10 + (1)(-x) + M_a \implies M_a = x - 10 \text{ kN-m}\end{aligned}\tag{4.100}$$

Section $b - b$:

$$\begin{aligned}\sum F_x &= 0 = N_b \\ \sum F_y &= 0 = 1 + 1 - V_b \implies V_b = 2 \text{ kN} \\ \sum M_{O_{b-b}} &= 0 = 10 + (1)(-x) + (1)(-(x-1)) + M_b \\ &\implies M_b = 2x - 11 \text{ kN-m}\end{aligned}\tag{4.101}$$

Section c – c:

$$\begin{aligned}
 \sum F_x &= 0 = N_c \\
 \sum F_y &= 0 = 1 + 1 - V_c \implies V_c = 2 \text{ kN} \\
 \sum M_{O_{c-c}} &= 0 = 10 + (1)(-x) + (1)(-(x-1)) - 3 + M_c \\
 &\implies M_c = 2x - 8 \text{ kN-m}
 \end{aligned} \tag{4.102}$$

Section d – d:

$$\begin{aligned}
 \sum F_x &= 0 = N_d \\
 \sum F_y &= 0 = 1 + 1 + \left(\int_3^x -1 ds \right) - V_d \implies V_d = 5 - x \text{ kN} \\
 \sum M_{O_{d-d}} &= 0 = 10 + (1)(-x) + (1)(-(x-1)) - 3 \\
 &\quad + \left(\int_3^x -1 ds \right) \left(-\frac{x-3}{2} \right) + M_d \\
 &\implies M_d = -\frac{x^2}{2} + 5x - 12.5 \text{ kN-m}
 \end{aligned} \tag{4.103}$$

The internal moment is fully-defined as a piecewise function along the length of the beam.

$$M(x) = \begin{cases} x - 10 \text{ kN-m} & 0 \leq x \leq 1 \text{ m} \\ 2x - 11 \text{ kN-m} & 1 \leq x < 2 \text{ m} \\ 2x - 8 \text{ kN-m} & 2 < x \leq 3 \text{ m} \\ -\frac{x^2}{2} + 5x - 12.5 \text{ kN-m} & 3 \leq x \leq 5 \text{ m} \end{cases} \tag{4.104}$$

Applying E-B beam theory:

$$\theta(x) = \frac{1}{EI} \begin{cases} \int M_1(x) dx \\ \int M_2(x) dx \\ \int M_3(x) dx \\ \int M_4(x) dx \end{cases} = \frac{1}{EI} \begin{cases} \frac{x^2}{2} - 10x + C_1 & 0 \leq x \leq 1 \text{ m} \\ x^2 - 11x + C_2 & 1 \leq x \leq 2 \text{ m} \\ x^2 - 8x + C_3 & 2 \leq x \leq 3 \text{ m} \\ -\frac{x^3}{6} + \frac{5x^2}{2} - 12.5x + C_4 & 3 \leq x \leq 5 \text{ m} \end{cases} \tag{4.105}$$

$$v(x) = \begin{cases} \int \theta_1(x) dx \\ \int \theta_2(x) dx \\ \int \theta_3(x) dx \\ \int \theta_4(x) dx \end{cases} = \frac{1}{EI} \begin{cases} \frac{x^3}{6} - 5x^2 + C_1x + C_5 & 0 \leq x \leq 1 \text{ m} \\ \frac{x^3}{3} - \frac{11x^2}{2} + C_2x + C_6 & 1 \leq x \leq 2 \text{ m} \\ \frac{x^3}{3} - 4x^2 + C_3x + C_7 & 2 \leq x \leq 3 \text{ m} \\ -\frac{x^4}{24} + \frac{5x^3}{6} - \frac{25x^2}{4} + C_4x + C_8 & 3 \leq x \leq 5 \text{ m} \end{cases} \tag{4.106}$$

There are eight constants of integration. Figure 4.41 reveals the following boundary and continuity conditions.

$$\begin{aligned}
 v(0) &= 0 & \theta(0) &= 0 \\
 v_1(1) &= v_2(1) & \theta_1(1) &= \theta_2(1) \\
 v_2(2) &= v_3(2) & \theta_2(2) &= \theta_3(2) \\
 v_3(3) &= v_4(3) & \theta_3(3) &= \theta_4(3)
 \end{aligned}
 \tag{4.107}$$

Thus providing eight equations for eight unknowns. Substituting the conditions (4.107) into (4.105) and (4.106) and solving:

$$\begin{aligned}
 C_1 &= 0 & C_2 &= \frac{1}{2} \\
 C_3 &= -\frac{11}{2} & C_4 &= -1 \\
 C_5 &= 0 & C_6 &= -\frac{1}{6} \\
 C_7 &= \frac{35}{6} & C_8 &= \frac{59}{24}
 \end{aligned}
 \tag{4.108}$$

These constants may be substituted into equations (4.105) and (4.106) to generate the slope and deflection curves. As a simple check on our work, plot both curves to ensure that the continuity conditions are satisfied at the interface of each piece of the functions.

Euler-Bernoulli theory is quite readily used to predict and understand the deformation of beams. One must always remember, however, that the assumptions and restrictions used to derive the theory must be maintained and valid. As mentioned previously, this deformation theory depends upon the assumption that cross-sections remain perpendicular to the neutral axis at all times throughout the beam. This implies that there is no shear deformation which, as discussed with regards to the flexure formula in Section 4.4, is not commonly true. It is critical that a user of the theory ensure that the beam is in fact long, slender, and deep to mitigate inaccuracies that may occur from this.

4.8 Beam Tables

It can be seen that the Euler-Bernoulli beam theory may be broadly applied to many physical problems. Its use is straightforward, and as long as the basic assumptions and restrictions are satisfied, it will yield accurate descriptions of the deflection of a beam under transverse load. In problems with a single load, such as Example 4.9, solution of the deflection curve is direct. However, as seen in Example 4.10, it can be quite tedious to construct the moment equation,

carry out all the integration, and solve the resulting algebraic system, especially if the beam is subjected to complex loading.

Fortunately, it is possible to construct complex solutions from simple cases. As discussed in detail in Section 2.6, as long as the deformation is small and relationships are linear transformations, a problem may be decomposed into simpler components which may then be solved individually. The superposition of these component solutions yields the complete solution. As it so happens, many of these simple solutions have been completed in standard forms and compiled into what are commonly called **beam tables**. It is a simple task to reference these tables to extract solutions to a wide variety of beam problems.

Some care should be taken when using beam tables. Not all tables are formatted identically. For example, if the table shows a force vector P pointed downward, then references that value of P in the expression for the deflection, it is considered positive. In other words, the sign convention given in the schematic in the table may not be what you are expecting. Additionally, tables are often organized by support type. The same loading conditions may be shown in multiple tables; take care to use the correct option for the specific problem at hand.

Beam tables are not universal. Many collections of tables will provide different information (deflection vs. slope vs. internal loads), different load and support conditions, etc. None are complete. While they can be combined in infinite different arrangements, there will always be a situation that is not accounted for in a given table. In those circumstances, it is necessary to return to fundamentals and solve the problem directly using Euler-Bernoulli beam theory. In such a situation, it is possible to construct your own beam table; simply use generic forces (P , w) and distances (a , b , L) when integrating to generate the deflection curve.

A short collection of beam tables may be found in Appendix C.

Example 4.11 (Beam Deflection by Tables).

PROBLEM STATEMENT:

Determine the deflection curve of the beam in Example 4.10, shown in Figure 4.40. The beam is steel with a rectangular cross-section 0.5 m deep and 0.1 m wide.

KNOWNs:

- Beam is cantilevered
 - x - and y -forces at support
 - Moment at support
- Length of 5 m

- Point force $P = 1$ kN 1 m from support
- Applied moment $M = -3$ kN-m 2 m from support
- Uniform distributed load $w = -1$ kN/m over final two meters of beam
- Beam is rectangular and prismatic
 - $h = 0.5$ m
 - $b = 0.1$ m
- Steel: $E = 200$ GPa (from Appendix B)

ASSUMPTIONS:

- Beam is in static equilibrium
- Neglect shear strain
- Small deformation

ANALYSIS:

As discussed in Example 4.10, this problem satisfies the requirements to use *Euler-Bernoulli beam theory*. This is critical even when using tables to solve; the tables are generated through the use of EB theory. The first step in using tables and superposition to solve the problem is to decompose the beam into its simplest components. As shown in Figure 4.42, this beam may be divided into three problems.

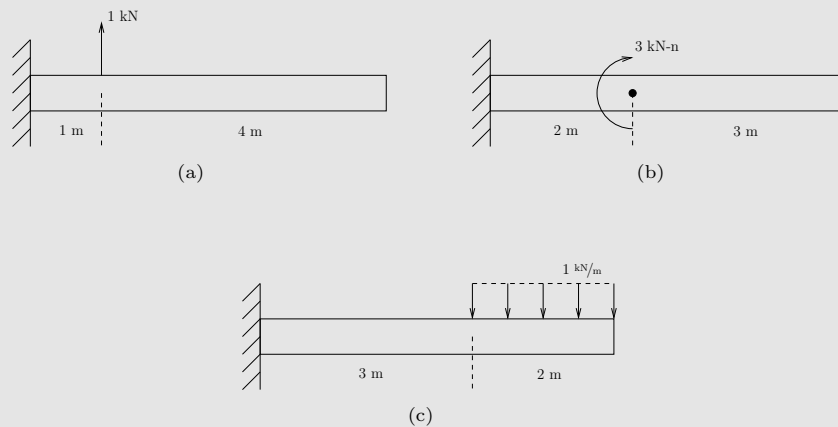


Figure 4.42: Decomposed Beam Problem

At this point it has been standard procedure to draw the free-body diagrams, determine external reactions, and find internal forces and moments through either shear and moment diagrams or static equilibrium of appropriate sections. However, when using beam tables, none of this is necessary; the results included in the tables have already accounted for all of those details. Therefore it is only necessary to relate the decomposed problems to their appropriate tables.

- Figure 4.42(a) → Appendix C.2
- Figure 4.42(b) → Appendix C.1
- Figure 4.42(c) → Appendix C.3

Decomposed Problem (a):

Appendix C.2 shows a beam cantilevered at the left end with a point load P directed upward (defining the positive direction). The force is located a units from the left edge and the total length is L . For the beam in Figure 4.42(a) (converting to base units for convenience), $P = 1000$ N, $a = 1$ m, and $L = 5$ m. Therefore the deflection of the beam due to the point force is:

$$\begin{aligned} v_P(x) &= \begin{cases} \frac{Px^2}{6EI}(3a-x) & x \leq a \\ \frac{Pa^2}{6EI}(3x-a) & x \geq a \end{cases} \\ &= \begin{cases} (8 \times 10^{-7})(3x^2 - x^3) \text{ m} & x \leq 1 \text{ m} \\ (8 \times 10^{-7})(3x - 1) \text{ m} & x \geq 1 \text{ m} \end{cases} \end{aligned} \quad (4.109)$$

Decomposed Problem (b):

Appendix C.1 shows a beam cantilevered at the left end with a moment M_a directed counterclockwise (defining the positive direction). The moment is located a units from the left edge and the total length is L . For the beam in Figure 4.42(b) (converting to base units for convenience), $M_a = -3000$ N-m, $a = 2$ m, and $L = 5$ m. Therefore the deflection of the beam due to the moment is:

$$\begin{aligned} v_M(x) &= \begin{cases} \frac{M_a x^2}{2EI} & x \leq a \\ \frac{M_a a}{2EI}(2x-a) & x \geq a \end{cases} \\ &= \begin{cases} (-7.2 \times 10^{-6})x^2 \text{ m} & x \leq 2 \text{ m} \\ (-1.44 \times 10^{-5})(2x-2) \text{ m} & x \geq 2 \text{ m} \end{cases} \end{aligned} \quad (4.110)$$

Decomposed Problem (c):

Appendix C.3 shows a beam cantilevered at the left end with a uniform distributed load w directed upward (defining the positive direction). The force begins a units from the left edge and terminates at the right edge. The

total length is L . For the beam in Figure 4.42(c) (converting to base units for convenience), $w = -1000$ N/m, $a = 3$ m, and $L = 5$ m. Therefore the deflection of the beam due to the moment is:

$$v_w(x) = \begin{cases} \frac{w(L-a)x^2}{12EI}(3L + 3a - 2x) & x \leq a \\ \frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2 - 4a^3x + a^4) & x \geq a \end{cases} \quad (4.111)$$

$$= \begin{cases} (-2.4 \times 10^{-6})(12x^2 - x^3) \text{ m} & x \leq 3 \text{ m} \\ (-2 \times 10^{-7})(x^4 - 20x^3 + 150x^2 - 108x + 81) \text{ m} & x \geq 3 \text{ m} \end{cases}$$

With the deflection of the beam due to each individual load determined, it is simply a matter of applying superposition to determine the total deformation field. First, a quick check of the magnitudes of the deflections in equations (4.109) – (4.111) shows that all three cases experience small deformation, so superposition is permissible.

Equations (4.109) – (4.111) are each piecewise-defined over unique regions. When superpositioning them, care must be taken to combine the appropriate expressions in each spatial region. The original beam schematic shown in Figure 4.38 clearly delineates four different portions of the beam, separated by applied loads. The first region of the beam extends from $x = 0$ to $x = 1$, so each expression for deflection that holds in this range should be summed.

$$\begin{aligned} v(x) &= v_{P1} + v_{M1} + v_{w1} \\ &= (8 \times 10^{-7})(3x^2 - x^3) + (-7.2 \times 10^{-6})x^2 + (-2.4 \times 10^{-6})(12x^2 - x^3) \\ &= (1.6x^3 - 33.6x^2) \times 10^{-6} \text{ m} \end{aligned} \quad (4.112)$$

The second region of the beam extends from $x = 1$ to $x = 2$:

$$\begin{aligned} v(x) &= v_{P2} + v_{M1} + v_{w1} \\ &= (8 \times 10^{-7})(3x - 1) + (-7.2 \times 10^{-6})x^2 + (-2.4 \times 10^{-6})(12x^2 - x^3) \\ &= (24x^3 - 360x^2 + 24x - 8) \times 10^{-7} \text{ m} \end{aligned} \quad (4.113)$$

The third region of the beam extends from $x = 2$ to $x = 3$:

$$\begin{aligned} v(x) &= v_{P2} + v_{M2} + v_{w1} \\ &= (8 \times 10^{-7})(3x - 1) + (-1.44 \times 10^{-5})(2x - 2) + (-2.4 \times 10^{-6})(12x^2 - x^3) \\ &= (2.4x^3 - 28.8x^2 - 26.4x + 28) \times 10^{-6} \text{ m} \end{aligned} \quad (4.114)$$

The final segment of the beam extends from $x = 3$ to $x = 5$:

$$\begin{aligned}
 v(x) &= v_{P2} + v_{M2} + v_{w2} \\
 &= (8 \times 10^{-7})(3x - 1) + (-1.44 \times 10^{-5})(2x - 2) \\
 &\quad + (-2 \times 10^{-7})(x^4 - 20x^3 + 150x^2 - 108x + 81) \\
 &= (-2x^4 + 40x^3 - 300x^2 - 48x + 118) \times 10^{-7} \text{ m}
 \end{aligned} \tag{4.115}$$

Thus the deflection of the beam is fully defined as (in meters):

$$v(x) = \begin{cases} (1.6x^3 - 33.6x^2) \times 10^{-6} & 0 \leq x \leq 1 \\ (24x^3 - 360x^2 + 24x - 8) \times 10^{-7} & 1 \leq x \leq 2 \\ (2.4x^3 - 28.8x^2 - 26.4x + 28) \times 10^{-6} & 2 \leq x \leq 3 \\ (-2x^4 + 40x^3 - 300x^2 - 48x + 118) \times 10^{-7} & 3 \leq x \leq 5 \end{cases} \tag{4.116}$$

DISCUSSION:

A cursory inspection shows that continuity is satisfied between each segment of the deflection curve. Furthermore, the deflection in equation (4.116) precisely matches that of Example 4.10. This is to be expected as the tables used to solve this problem are generated using Euler-Bernoulli beam theory. Tables may be found for many beam problems, and created for any. In many cases tables can be a more efficient tool for determining the deflection of beams. Additionally, although the tables in Appendix C do not include it, many tables also provide internal moment distributions or shear and moment diagrams, allowing for quick determination of stresses as well.

Care must always be taken that tables are used appropriately. The table must exactly match the problem at hand. Boundary conditions, supports, and load types and locations are critical. When superpositioning results of individual loads, both the magnitude of the deformation and the regions for which each description hold must be confirmed.

As mentioned previously, some tables are quite extensive and the principle of superposition allows them to be combined freely. Superposition may also be used when solving for the deflection directly using Euler-Bernoulli beam theory. Consideration of individual loads may lead to simpler problems that can later be combined into the complete deflection field.

Some older engineers may be especially familiar with three specific beam problems (in which θ and δ refer to the slope and deflection of the end of the beam, respectively):

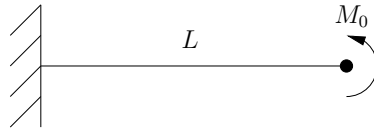


Figure 4.43: Cantilevered beam with applied moment

$$\theta = \frac{M_0 L}{EI}$$

$$\delta = \frac{M_0 L^2}{2EI}$$

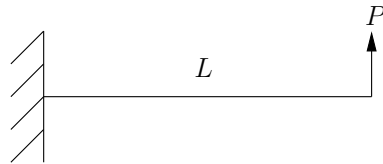


Figure 4.44: Cantilevered beam with point force

$$\theta = \frac{PL^2}{2EI}$$

$$\delta = \frac{PL^3}{3EI}$$

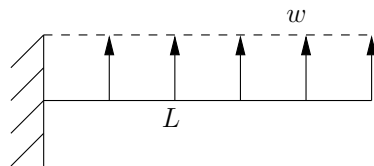


Figure 4.45: Cantilevered beam with uniform distributed force

$$\theta = \frac{wL^3}{6EI}$$

$$\delta = \frac{wL^4}{8EI}$$

These six relations have been called the **myosotis equations**, after the genus of flowers more colloquially known as *forget-me-nots*. Naming puns aside, these equations were at one time considered critical for an engineer to commit to memory, and saved many, many man-hours of time in longhand calculations. However, times change and progress marches inexorably onward. What were once one of the most valuable tools available to us are now little more than a historical note.

More broadly, beam tables in general see increasingly limited use in modern application. As discussed several times elsewhere in this text, the advent of modern computing technology has simply rendered them obsolete. That being said, they are still a valuable learning tool, allowing students to quickly see how different loads might affect beam deformation as well as gaining additional experience in applying superposition to increasingly complex problems, which still has immense practical value.

4.9 Chapter Summary

Beams are the third surrogate model typically considered in solid mechanics. While geometrically similar to bars and shafts, the introduction of transverse deflection creates significant differences between this model and those presented

earlier in the book. Once again, this model (as presented in this chapter) is only applicable when the following assumptions and restrictions hold:

1. The material is continuous, homogeneous, isotropic, and linearly elastic
2. The structure is long, slender, and straight
3. The member undergoes only small deformation
4. The beam is symmetric about the y -axis when using the coordinate system defined in this chapter
5. The structure is loaded by moments parallel to the z -axis, or by forces that create such moments
6. The moments are approximately centered on the y -axis of the member
7. Area changes occur gradually
8. Shear strains in the xy -plane are neglected

As with bars and shafts, beams are geometrically one-dimensional along the longitudinal axis, allowing for the physical description of the problem to reduce to the neutral surface with cross-sectional properties introduced as parameters. Deflection occurs off-axis, but with the assumption of small deformation is restricted to the transverse direction with no longitudinal components. As it varies only with location along the length of the beam, it is considered quasi-1D (i.e., v occurs in the y -direction but is only a function of x). Deformation of the beam in the transverse direction may be measured by either the curvature of the beam or the vertical deflection.

Stresses in the beam are due primarily to bending action. Bending stress is a normal stress oriented parallel to the x -axis and varies linearly with distance from the neutral surface (again, if shear strain is neglected). This stress in many cases dominates the loading of a structure. If the internal moment in a beam is created by a transverse force, it also carries a shear component of the internal force which necessitates the existence of shear stress. This shear stress is typically significantly smaller than the normal bending stress, but that is not always the case.

An important design factor of beams the shape and size of the cross-section. Normal bending stress is maximized at the furthest point from the neutral axis while the shear stress is zero at that location. Conversely, shear stress achieves its maximum value at the neutral axis where the normal stress does not exist. For this reason, the outer regions of beams are designed to support high bending stress while the center regions are designed to bear more shear load. The second area moment I , which describes distribution of material about the neutral surface, is the primary (although not the only) geometric factor in beam design. Distributing material further from the neutral axis directly increases the stiffness of the beam and reduces stress.

Both the flexure formula for normal stresses and Euler-Bernoulli beam theory for deflection depend upon the condition that shear strains be zero. In many, if not most, real situations, moments are created by transverse forces. These transverse forces result in shear stresses which must be coupled to shear strains. Neglecting shear strain (i.e., assuming cross-sections of the beam remain parallel to the neutral axis at all times) is a major assumption which is immediately flawed in most cases. Use of beam theory therefore often includes an acceptance of error inherent in the analysis. It is the responsibility of the engineer to decide if the error is acceptable.

While the work presented in this chapter makes use of multiple restrictions, it is possible to solve beams that do not fit into those limits. Beams that are not symmetric about the y -axis can be analyzed (to an extent) through creative decomposition of the moment into components and determination of the neutral axis, although these analyses often neglect the more exotic deformation that may occur. Curved beams can be effectively analyzed through the use of strain energy, using methods such as *Castigliano's theorem* or the *principle of virtual work*. Finally, many approaches have been developed to address the issue of neglecting shear strains. Most of these methods rely upon making different assumptions about the deformation of the cross-section. The first of these methods to be developed, and the most commonly used, is the **Timoshenko beam theory**. In Timoshenko beam theory, unlike Euler-Bernoulli beam theory, the cross-sections of the beam are allowed to rotate about the neutral axis (but not deform), introducing an additional variable Ψ to the deformation field. In the following years, additional *higher order shear theories* have been developed which introduce less limited assumptions (e.g., the cross-sections deform parabolically). These are improvements, but in all cases depend upon a relatively arbitrary assumption, which makes beam theories particularly difficult to reconcile with the complete three-dimension continuum model.

4.9.1 List of Symbols

M :	Internal moment
ρ :	Radius of curvature
κ :	Curvature
ε_{xx} :	Normal strain
σ_{xx} :	Normal (bending) stress
E :	Young's modulus
\bar{y} :	Centroid of the cross-section
y :	Distance from the neutral surface
I :	Second area moment, also known as the area moment of inertia
τ_{xy} :	Transverse shear stress
γ_{xy} :	Shear strain
b :	Material thickness at a location
Q :	First area moment
V :	Shear force
w :	Distributed load
θ :	Slope of deformed beam
v :	Deflection of beam

Chapter 5

Two-Dimensional Loading

Axial deformation was explored in Chapter 2, torsion in Chapter 3, and bending in Chapter 4. These surrogate models each provide a reduction of the physics to a one-dimensional case. While useful and relatively simple, these models are quite limited in their scope. At this point it is convenient for us to consider more complex loading conditions and begin treating two-dimensional problems. While there are many examples of 2D surrogates, the bulk of this chapter will focus on **plane stress**, with some discussion of **plane strain** at the end.

5.1 Fundamentals of Plane Stress Models

Because 2D models are not simplified as much from the continuum model, fewer basic assumptions are required for plane stress.

1. The material is homogeneous, isotropic, continuous, and linearly elastic
2. There are no out-of-plane stresses

The first assumption is obvious based on all work completed so far. The second assumption also seems obvious, given the name of the model, but warrants further investigation. Intuitively, we can understand that given a thin, flat plate with forces only applied laterally (i.e., no forces through the thickness), stresses would only develop within that plane, as shown in Figure 5.1.

While this is not incorrect, it is a limited view of plane stress. At this point no restrictions have been made on the geometry of the material. It could be an entirely arbitrary 3D solid. The only limitation we have introduced is that all stresses exist in the same plane. At this point it becomes critical to better define what is meant by *all stresses*. From equation (1.57) and the discussion in Section 1.2.1, stresses are defined in the pointwise sense. That is, each material particle experiences its own unique stress. In other words, if we consider individual particles rather than the entire structure, plane stress can be considered much

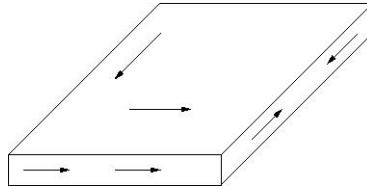


Figure 5.1: A thin plate in a state of plane stress

more broadly. As shown for the shaft in Figure 5.2 (and recalling the appropriate material from Chapter 3), particle A is subjected to shear stress τ_{xy} in the xy -plane while particle B is subjected to stresses in the xz -plane. The entire structure has 3D stresses, but each of the particles experiences stress in a single plane.

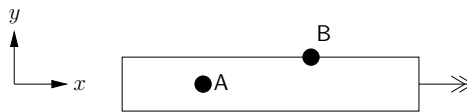


Figure 5.2: Plane stress within three-dimensional stress

With this in mind it is useful to define the **stress element**. A stress element is a frame used to represent stresses on a material point. It is not the particle itself, and changes to a stress element do not imply changes to the material. It is a mathematical tool that is helpful in defining coordinate systems and components of stresses for individual particles, nothing more. In fact, we have already seen stress elements; Figure 1.21 is a good example. While the shape of the element is arbitrary, square (or cubic) elements are often useful. Figure 5.3(a) shows a fully-stressed 3D cubic element. If the material point experiences plane stress, the element simplifies to that shown in Figure 5.3(b). Most often, plane stress is shown on a 2D stress element as in Figure 5.3(c). Figures 5.3(b) and 5.3(c) show the same state of stress; (c) is simply rotated so that the z -direction is oriented directly out of the page.

With these considerations, we may realize that each surrogate model considered so far can be another example of a plane stress problem. The stresses on the shaft, for example, are not confined to a single plane (Figure 5.2). On the lateral surface the shear stresses point in the x - and y -directions, but on the top of the shaft the stresses are oriented in the x - and z -directions. The shaft is clearly subjected to a 3D stress state. However, the element on the left face experiences stress only in the xy -plane and the element on the top only shows stresses in the xz -plane. Each of these elements exists in a state of plane stress. Bars and beams are even more direct in that the entire structure experiences 1D or 2D stress.

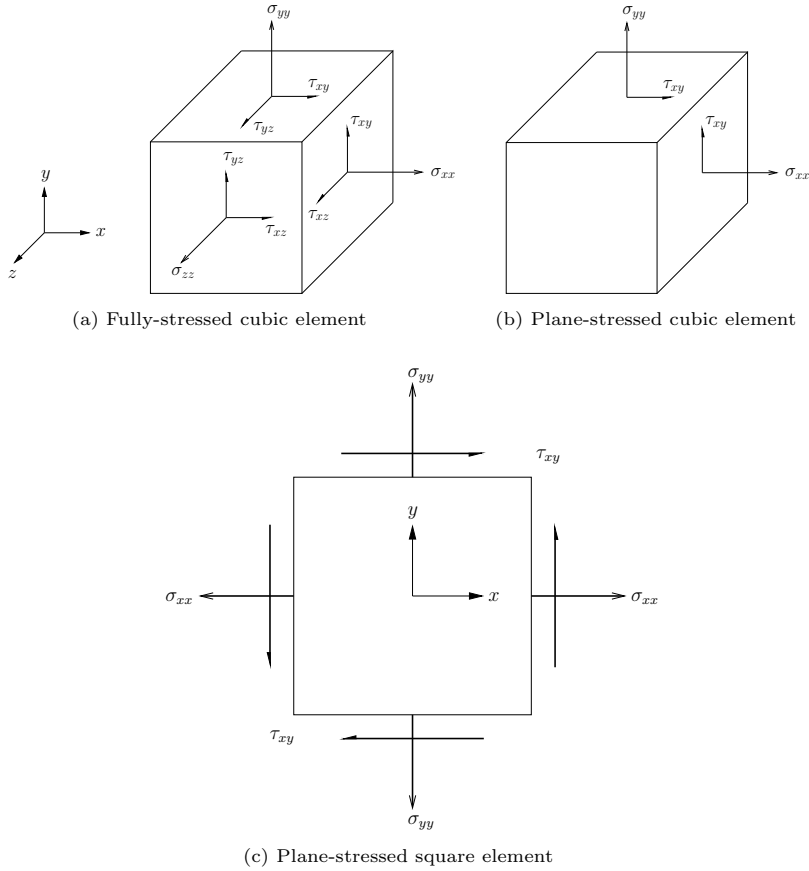


Figure 5.3: Stress elements

5.1.1 Hooke's Law for Plane Stress

Consider equation (1.81), the generalized Hooke's law for three-dimensional solids (repeated below).

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad (5.1)$$

In a state of plane stress, all nonplanar stresses must be zero. Assuming our working plane is the xy -plane, this implies that all z -components of the stress

must vanish.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 0 \\ \tau_{xy} \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad (5.2)$$

Immediately, the shear strains γ_{yz} and γ_{xz} must also be zero. From the third row of (5.2):

$$\varepsilon_{zz} = \frac{\nu}{\nu-1}(\varepsilon_{xx} + \varepsilon_{yy}) \quad (5.3)$$

Substituting (5.3) into the first two rows of (5.2) and simplifying, we find **Hooke's law for plane stress**.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad (5.4)$$

Note in equation (5.4) that only x - and y -stresses are present. All z -stresses are set to zero. Take some care however in applying the same logic to the strains. From equation (5.3), ε_{zz} is clearly not zero. This constitutive theory does not imply plane strain. The normal z -strain still exists, but is completely deterministic from the x - and y -strains through the Poisson effect.

It can be useful in some cases to consider strains as a function of stress. For such situations, equation (5.4) can be solved for the strains, resulting in:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \quad (5.5)$$

5.2 Plane Stress Transformations

Consider the simple bar problem shown in Figure 5.4(a). The bar has uniform cross-sectional area and is loaded by a single tensile point force P at either end. Taking a section $b-b$ (as shown in Figure 5.4(b)), the internal normal force is found to be $N_b = P$ and the normal stress is $\sigma_b = P/A$. The free-body diagram in Figure 5.4(b) included internal shear force for completeness, although it is normally excluded in free-body diagrams for bars because static equilibrium requires it (as well as shear stress) be zero.

This result is well-established and understood from the material in Chapter 2. However, careful thought may lead us to ask why we drew the section in the way we did. Is there any requirement that it be perpendicular to the axis of the bar? What would happen if it were not? It is simple enough to investigate. Considering the section $c-c$ shown in Figure 5.4(c), we see the internal face of the section oriented at 45° to the longitudinal axis. This section behaves the

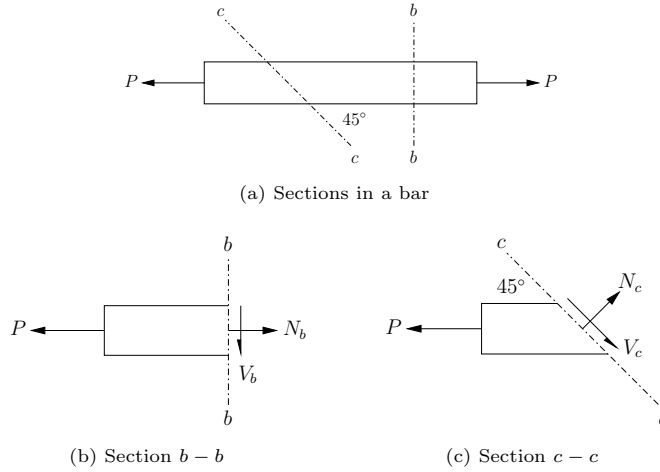


Figure 5.4: Internal force and stress in a bar

same as any other; it experiences internal forces normal and tangential to its surface. Applying static equilibrium to the section, we find:

$$\begin{aligned}\sum F_x = 0 &= N_c \sin(45^\circ) + V_c \cos(45^\circ) - P \\ \sum F_y = 0 &= N_c \cos(45^\circ) - V_c \sin(45^\circ)\end{aligned}\quad (5.6)$$

$$N_c = V_c = \frac{P}{\sqrt{2}}$$

These are clearly not the same as the internal forces found using section $b-b$. Assuming the bar has a rectangular cross-section with height h and thickness t , the area of section $b-b$ is:

$$A_b = ht \quad (5.7)$$

and the area of section $c-c$ is:

$$A_c = \left(\frac{h}{\sin(45^\circ)} \right) t = ht\sqrt{2} \quad (5.8)$$

Thus, the normal stress at section $c-c$ can be found.

$$\sigma_c = \frac{N_c}{A_c} = \frac{P/\sqrt{2}}{ht\sqrt{2}} = \frac{P}{2ht} = \frac{P}{2A_b} = \frac{\sigma_b}{2} \quad (5.9)$$

Likewise, the shear force V_c induces a shear stress.

$$\tau_c = -\frac{\sigma_b}{2} \quad (5.10)$$

The stress (and internal forces) found using section $b-b$ and section $c-c$ are completely different. In the first case, force divided by area yields an isolated

normal stress. In the second case, not only has the value of the normal stress changed, a shear stress has also appeared. How is this possible? How can the state of stress in the bar change if the external loading and geometry of the bar are the same?

The short answer is that they cannot. Nothing about the bar has changed. Why then do we see this variation in results? What has changed is not the bar, but *our perspective*. The sections taken in the material are nothing more than examples of stress elements, frames set by us to define the coordinates we choose to use. In Figure 5.4(b) the edges of the stress element are parallel to the bar. In Figure 5.4(c) the stress element (not the material!) is rotated such that its edges are 45° to the axis of the bar. This change in our point of view allows us to see the stresses differently. The stress has not changed, we have simply decomposed it into different components.

Rotation of the stress element is known as a **coordinate transformation**. Consider the point P shown in Figure 5.5. Measured using the (x, y) coordinate system, P is in the first quadrant, say around $(3, 4)$. If instead we measure P using the rotated coordinate system (x', y') , then P is in the fourth quadrant, somewhere around $(4.6, -2)$. P has not moved at all, we simply referenced it using different axes, so we see different coordinates. This is essentially what happens when a stress element is rotated.

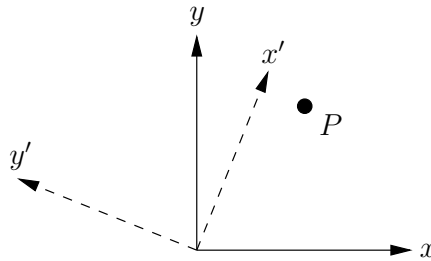


Figure 5.5: Rotation of coordinate axes

What purpose may this serve? Frankly, it's significantly simpler to find the stress σ_b from Figure 5.4 than it is to find σ_c . Why then bother with it? Simply put, if the representation of the stress changes with orientation of the stress element, then there may be some configuration in which we see the most critical stresses. For example, ductile materials are known to fail due to shear stresses. Our basic bar analysis only results in normal stress. Does this mean that all bars are indestructable? Hopefully we all realize this is not the case. Transformation of our coordinate system, as done in Figure 5.4(c), allows us to identify the shear stress that was hidden in our initial coordinate system. Likewise, there may be other stresses that warrant discovery.

With this understanding it is, in theory, possible to transform any stress element through some angle and determine how the stresses on the element change. Illustrated in Figure 5.6, a plane stress element oriented along the (x_1, y_1) coordinate system experiences stresses σ_{x_1} , σ_{y_1} , and τ_{xy_1} . If this element

is rotated through some angle θ to a new coordinate system (x_2, y_2) , then the total state of stress, though unchanged, is resolved into new components σ_{x2} , σ_{y2} , and τ_{xy2}

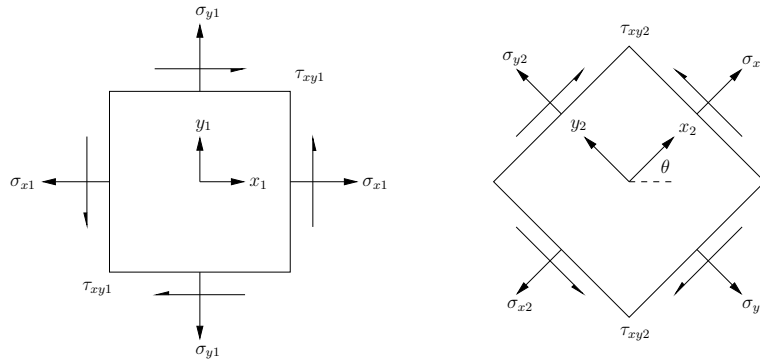


Figure 5.6: General plane stress element transformation

The primary difficulty arises in transforming the information contained in the first element in Figure 5.6 to the second. While it was straightforward enough to compute the stress at some arbitrary angle for the bar problem, that is not necessarily the case with all plane stress problems. Furthermore, it is a time consuming and tedious process to complete for every problem at every possible angle that may be a concern. Fortunately, the process may be generalized. Recall that stress elements are arbitrary frames of reference. We can choose a single element such that the information of both elements in Figure 5.6 is present. The right triangular element shown in Figure 5.7 is one such choice; the perpendicular edges of the triangle represent the initial (x_1, y_1) coordinate system while the oblique edge has normal and tangential components parallel to the (x_2, y_2) system.

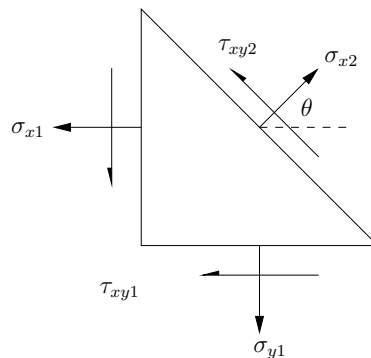


Figure 5.7: Right triangular stress element

A stress element is a representation of a single material point with very small

(i.e., zero) area. In other words, the limit in equation (1.57) has already been taken and stresses are precisely the force divided by the respective area. If A_{x1} is the area of the negative x_1 -face of the stress element (the left face), then a free-body diagram of the stress element may be drawn as in Figure 5.8. Note that while the shear stresses τ_{xy1} on the x_1 - and y_1 -faces are identical, those faces may have different areas dependent on θ so the shear forces on the two faces are not typically equivalent.

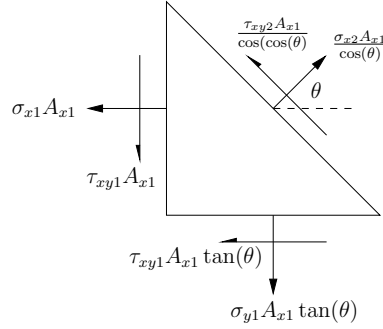


Figure 5.8: Free-body diagram of the stress element

It is now possible to apply force equilibrium. The details of the derivation are left as a homework problem, but following simplification and application of some trigonometric identities, the equilibrium equations reduce to:

$$\sigma_{x2}(\theta) = \frac{\sigma_{x1} + \sigma_{y1}}{2} + \frac{\sigma_{x1} - \sigma_{y1}}{2} \cos(2\theta) + \tau_{xy1} \sin(2\theta) \quad (5.11)$$

$$\tau_{xy2}(\theta) = -\frac{\sigma_{x1} - \sigma_{y1}}{2} \sin(2\theta) + \tau_{xy1} \cos(2\theta) \quad (5.12)$$

Thus the transformed stresses are given as functions of the transformation angle θ . Although it is not a part of the free-body diagram Figure 5.8, σ_{y2} may be found easily by recognizing that the y_2 -face is perpendicular to the x_2 -face.

$$\sigma_{y2}(\theta) = \sigma_{x2}(\theta + 90^\circ) = \frac{\sigma_{x1} + \sigma_{y1}}{2} - \frac{\sigma_{x1} - \sigma_{y1}}{2} \cos(2\theta) - \tau_{xy1} \sin(2\theta) \quad (5.13)$$

Equations (5.11) – (5.13) relate the stresses in any initial coordinate system to any transformed system for a plane stress problem. The notations (x_1 , x_2 , etc.) were chosen to highlight the fact that these transformations work for any coordinate systems as long as the angle of rotation is known. Because the initial stresses are most often defined in the (x , y) system, these equations are typically written as:

$$\begin{aligned} \sigma_{x1} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta) \\ \sigma_{y1} &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) - \tau_{xy} \sin(2\theta) \\ \tau_{xy1} &= -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta) \end{aligned} \quad (5.14)$$

These are known as the **plane stress transformation equations**, and hold for any plane stress problem. They are used to determine the state of stress in any rotated coordinate system. As a simple check, we can confirm that the following hold:

$$\begin{aligned}\sigma_{x1}(0) &= \sigma_x \\ \sigma_{x1}(90) &= \sigma_y \\ \tau_{xy1}(90) &= -\tau_{xy}\end{aligned}\tag{5.15}$$

Example 5.1 (Plane Stress Transformations).

PROBLEM STATEMENT:

Determine the state of stress for the bar shown in Figure 5.4(c) and draw both the initial and the rotated stress elements.

KNOWN:

- The state of stress in the (x, y) coordinate system is given by:

$$\begin{aligned}\sigma_x &= \frac{P}{A} \\ \sigma_y &= 0 \\ \tau_{xy} &= 0\end{aligned}\tag{5.16}$$

- The angle of rotation to the second configuration is 45°

ANALYSIS:

Typically, the first step of any analysis would be to confirm that the appropriate theory holds. In this case though, we are given a state of plane stress to begin the problem (for example, another engineer may have already completed the bar analysis), so it is not necessary this time to confirm that bar theory was an appropriate choice. A simple check of our data confirms that plane stress does hold though, as all stresses lie in the xy -plane.

To determine the transformed stress, substitute the initial values and the transformation angle into the transformation equations (5.14).

$$\begin{aligned}\sigma_{x1} &= \frac{\sigma_x + 0}{2} + \frac{\sigma_x - 0}{2} \cos(2(45^\circ)) + (0) \sin(2(45^\circ)) = \frac{\sigma_x}{2} \\ \sigma_{y1} &= \frac{\sigma_x + 0}{2} - \frac{\sigma_x - 0}{2} \cos(2(45^\circ)) - (0) \sin(2(45^\circ)) = \frac{\sigma_x}{2} \\ \tau_{xy1} &= -\frac{\sigma_x - 0}{2} \sin(2(45^\circ)) + (0) \cos(2(45^\circ)) = -\frac{\sigma_x}{2}\end{aligned}\tag{5.17}$$

This exactly matches the results of equations (5.9) and (5.10). The stress elements are shown in Figure 5.9.

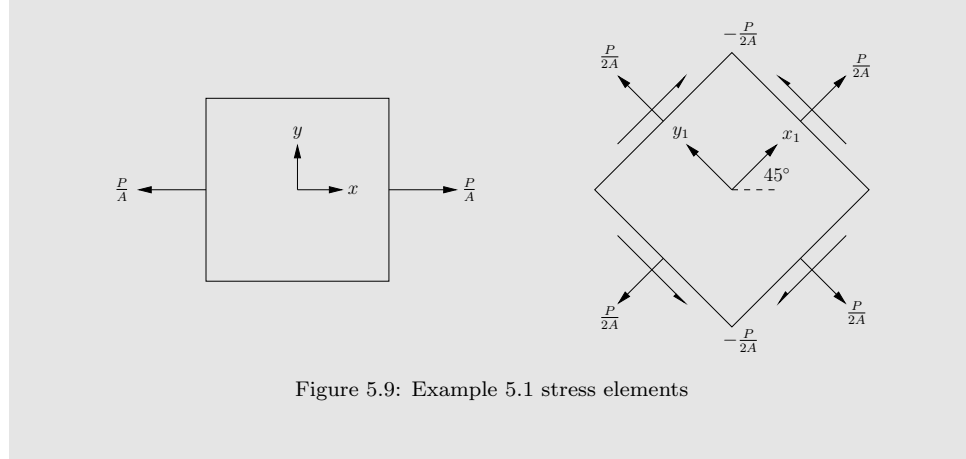


Figure 5.9: Example 5.1 stress elements

5.3 Principal and Maximum Shear Stresses

As mentioned in the previous section, one of the primary reasons stress transformations are used is to find stresses that may be critical to material failure. Intuitively, the 45° angle used in Example 5.1 is not special. Any nonzero value of θ used for a stress element would convert at least part of the normal stress into a shear stress, resulting in different values for both. In practical application, the “worst-case scenario” is of the most interest. In that case, it is enough to find the configuration that shows the maximum magnitude of the stress. Calculus provides all the necessary tools.

$$\frac{d\sigma_{x_1}}{d\theta} = 0 = \left(\frac{\sigma_x - \sigma_y}{2} \right) (-2 \sin(2\theta)) + \tau_{xy} (2 \cos(2\theta)) \quad (5.18)$$

Simplifying:

$$\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\sin(2\theta_p)}{\cos(2\theta_p)} = \tan(2\theta_p) \quad (5.19)$$

θ_p is the angle that satisfies equation (5.19), known as the **principal angle**. The tangent function is π -periodic, meaning that there are two values of θ_p between 0 and 180° that satisfy this equation (because θ_p is multiplied by two). Thus, one value corresponds to the maximum stress, and the other value corresponds to the minimum stress. Casual observation of the tangent function makes it obvious that these values of θ_p are 90° from each other. That is, the maximum stress is 90° from the minimum stress; in other words, at θ_p if σ_{x_1} is the maximum possible normal stress then σ_{y_1} must be the minimum, and both appear on the same stress element!

The presence of trigonometric functions suggests that a triangular representation of these results, such as the one shown in Figure 5.10, may be useful.

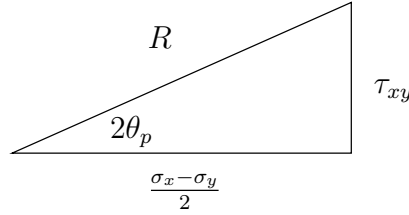


Figure 5.10: Graphical representation of equation (5.19)

From this schematic, it is straightforward to determine the hypotenuse R .

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (5.20)$$

With R defined, expressions for $\sin(2\theta_p)$ and $\cos(2\theta_p)$ are easily constructed.

$$\begin{aligned} \sin(2\theta_p) &= \frac{\tau_{xy}}{R} \\ \cos(2\theta_p) &= \frac{\sigma_x - \sigma_y}{2R} \end{aligned} \quad (5.21)$$

The expressions in equation (5.21) can be substituted directly into the stress transformation equations for σ_x and σ_y , (5.14), avoiding the need to solve for θ_p directly. Simplifying these expressions yields:

$$\begin{aligned} \sigma_{x1,p} = \sigma_{1p} &= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ \sigma_{y1,p} = \sigma_{2p} &= \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \end{aligned} \quad (5.22)$$

These are known as the **in-plane principal stresses** and represent the maximum (most tensile, σ_{1p}) and minimum (most compressive, σ_{2p}) normal stresses on an element. It is possible that both are positive or both are negative; the first principal stress would still be the most positive and the second principal stress the most negative.

Note what happens if the expressions for θ_p , (5.21), are substituted into the shear stress transformation equation in equation (5.14).

$$\tau_{xy1}(\theta_p) = -\frac{\sigma_x - \sigma_y}{2} \left(\frac{\tau_{xy}}{R}\right) + \tau_{xy} \left(\frac{\sigma_x - \sigma_y}{2R}\right) = 0 \quad (5.23)$$

There is no shear stress acting on the element at the principal angle. In other words, if an element shows no shear stress, then the normal stresses acting on it are the principal stresses.

Since the principal angle does not provide the maximum shear stress, it must be found independently. The same process if followed for the shear stress, beginning by taking the derivative of the shear stress transformation equation and

equating it to zero. The end result is an expression for the **in-plane maximum shear stress**.

$$\tau_{xy1,p} = \tau_{max,p} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (5.24)$$

This maximum shear stress occurs at 45° from the principal angle. There is no complementary minimum shear stress; rotating the element 90° results in the same shear stress with opposite direction. Note that there is no restriction on normal stresses in the maximum shear configuration; they are not required to be zero the way shear stress is in the principal configuration.

Equations (5.22) and (5.24) have some obvious connections. It is immediately clear that the radical appears in all three expressions. Simple algebraic manipulation gives explicit relations.

$$\begin{aligned} \tau_{max,p} &= \frac{\sigma_{1p} - \sigma_{2p}}{2} \\ \sigma_{1p,2p} &= \frac{\sigma_x + \sigma_y}{2} \pm \tau_{max,p} \end{aligned} \quad (5.25)$$

Example 5.2 (Principal and Maximum Shear Stresses).

PROBLEM STATEMENT:

Determine the in-plane principal and maximum shear stresses for the bar shown in Figures 5.4.

KNOWNs:

- The state of stress in the (x, y) coordinate system is given by:

$$\begin{aligned} \sigma_x &= \frac{P}{A} \\ \sigma_y &= 0 \\ \tau_{xy} &= 0 \end{aligned} \quad (5.26)$$

ANALYSIS:

As in Example 5.1, we are given a state of plane stress to begin the problem, so it is not necessary to confirm that bar theory was an appropriate choice. A simple check of our data confirms that plane stress does hold though, as all stresses lie in the xy -plane. Additionally, since the stress in the bar is uniform at all locations, it does not matter where we choose our stress element.

To determine the principal and maximum shear stresses, begin by drawing the stress element.

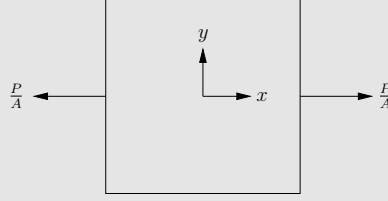


Figure 5.11: Initial stress element

Because there are no shear stresses acting on this element, the principal stresses must be the normal stresses already present. However, we will confirm this by substituting the xy -stresses into the principal stress equations (5.22):

$$\begin{aligned}\sigma_{1p} &= \frac{P/A + 0}{2} + \sqrt{\left(\frac{P/A - 0}{2}\right)^2 + 0^2} = \frac{P}{A} \\ \sigma_{2p} &= \frac{P/A + 0}{2} - \sqrt{\left(\frac{P/A - 0}{2}\right)^2 + 0^2} = 0\end{aligned}\quad (5.27)$$

The principal angle is therefore $\theta_p = 0^\circ$.

To determine the in-plane maximum shear stress, we have three options:

1. Substitute xy -stresses into the maximum shear equation (5.24)
2. Substitute the principal stresses into equation (5.25)
3. Add 45° to the principal angle, then substitute that value into the stress transformation equations

We will solve for the shear stress using both methods 1 and 2 to confirm their validity. Beginning with equation (5.24):

$$\tau_{max,p} = \sqrt{\left(\frac{P/A - 0}{2}\right)^2 + 0^2} = \frac{P}{2A}\quad (5.28)$$

And then working through equation (5.25):

$$\tau_{max,p} = \frac{P/A - 0}{2} = \frac{P}{2A}\quad (5.29)$$

Thus, both the in-plane principal stresses and the maximum shear stress for the bar are known. The respective stress elements are shown in Figure 5.12.

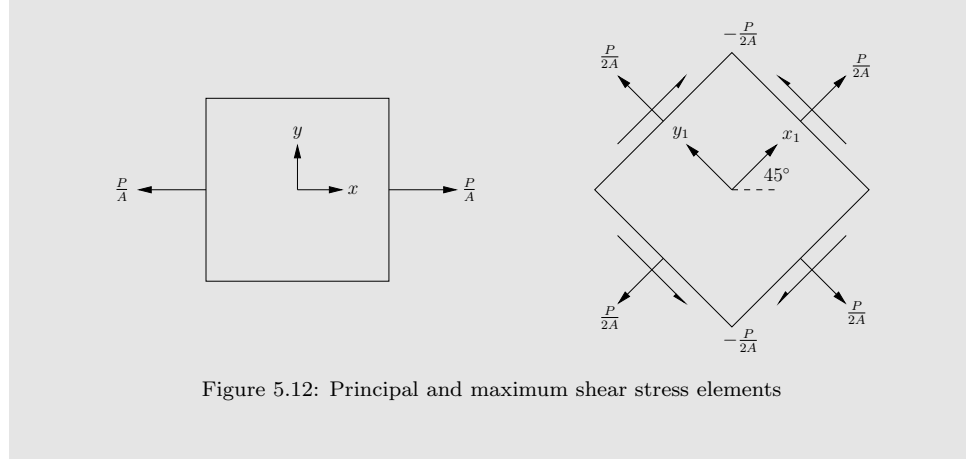


Figure 5.12: Principal and maximum shear stress elements

5.3.1 Triaxial Stress

Equations (5.22) define the in-plane principal stresses for a plane stress problem. Emphasis has been made on the language *in-plane*. An important point to keep in mind is that although the z -stress is zero, it still exists. When determining principal stresses, it is entirely possible that a zero stress could be greater or less than the two in-plane stresses. For that reason, discussion of principal stresses must include all three terms. A material subjected to normal stress in all three directions and no shear stress is said to be in a state of **triaxial stress**.

This text has used the notations σ_{1p} and σ_{2p} to refer to the in-plane principal stresses, rather than simply σ_1 and σ_2 . This is because a third stress, $\sigma = 0$, could change the order of most tensile to most compressive. When considering all three principal stresses, the convention remains the same, with σ_1 being the most positive and σ_3 being the most negative; σ_2 is the intermediate.

As an example, consider the plane stress problem in Figure 5.13(a). The in-plane principal stresses are known, and convert to the triaxial stresses as shown:

$$\begin{array}{l} \sigma_{1p} = 20 \\ \sigma_{2p} = -40 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \sigma_1 = 20 \\ \sigma_2 = 0 \\ \sigma_3 = -40 \end{array} \quad (5.30)$$

Alternatively, the plane stress problem in Figure 5.13(b) leads to the following triaxial stresses.

$$\begin{array}{l} \sigma_{1p} = 50 \\ \sigma_{2p} = 40 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \sigma_1 = 50 \\ \sigma_2 = 40 \\ \sigma_3 = 0 \end{array} \quad (5.31)$$

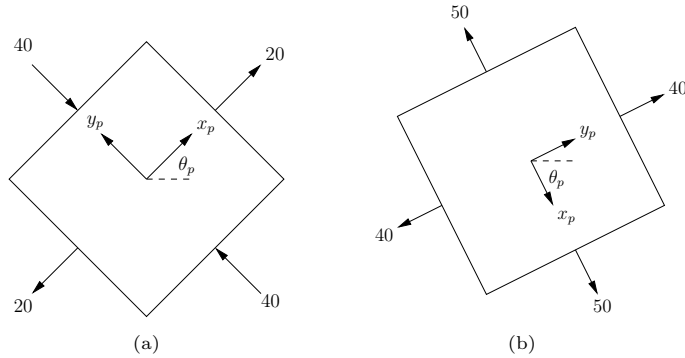


Figure 5.13: In-plane principal stresses

In the first case, σ_{2p} became σ_3 . In the second case, σ_{1p} and σ_{2p} remained in the same order with the addition of a new σ_3 . The use of σ_1 and σ_2 for σ_{1p} and σ_{2p} in situations like that in Figure 5.13 is common, but can be misleading. In general, principal stresses should be reported as triaxial stresses to avoid confusion.

5.4 Mohr's Circle

Consider the plane stress transformation equations (5.14). Obviously these equations relate stresses in the (x_1, y_1) system to those in the (x, y) system through some angle θ . Viewed geometrically though, they provide a parametric representation of a circle. Points on this circle, known as **Mohr's circle for plane stress**, are the states of stress at various angles.

A sample circle is shown in Figure 5.14. It is important to keep in mind how the transformation equations were derived to understand how to construct and interpret Mohr's circle. Looking back at Figure 5.7, a single face of the stress element was rotated through some angle. Intuitively, this means the stresses on that face should be the ones considered, a single normal stress and a single shear stress (σ, τ) .

- The axes for the circle are normal stress σ and shear stress τ .
- A stress element returns to its initial configuration after rotating 180° . Since the circle has 360° , angles in the circle must be 2θ .
- If the stress element is rotated 90° , then σ_x becomes σ_y and τ_{xy} becomes $-\tau_{xy}$, therefore the points (σ_x, τ_{xy}) and $(\sigma_y, -\tau_{xy})$ must be on opposite sides of the circle from each other.
- As a matter of convention and convenience, the τ -axis is drawn positive downward. This results in angles θ being positive counterclockwise.

- If σ_x and σ_y are on opposite sides of the circle, then the center of the circle must be:

$$C = \frac{\sigma_x + \sigma_y}{2} \quad (5.32)$$

- Using the center and one of the points on the circle, the radius can be defined as:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (5.33)$$

- Since the shear stress is zero in the principal configuration, the in-plane principal stresses are easily identified as the leftmost and rightmost points on the circle, on the σ -axis.
- The maximum shear stress occurs at the top and bottom surface of the circle.

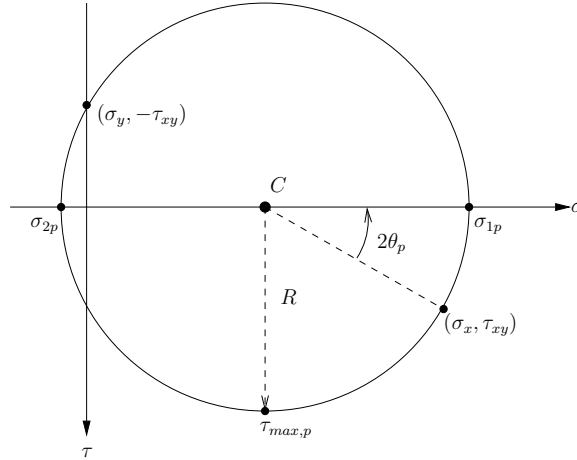


Figure 5.14: Mohr's circle for plane stress

We can clearly see the connection between Mohr's circle and the plane stress transformation equations now. The maximum shear stress is simply the radius of the circle (hence the use of R in Figure 5.10). Adding or subtracting the radius to the center of the circle yields the principal stresses, and mathematically identical expressions to (5.22).

Construction and use of Mohr's circle is direct once the initial state of xy -stress is known, with steps outlined in Table 5.1.

Table 5.1: Steps to construct Mohr's circle

-
1. Draw the axes, keeping in mind that the τ -axis is positive downward.
 2. Plot the points (σ_x, τ_{xy}) and $(\sigma_y, -\tau_{xy})$.
 3. Connect the points by a line. This line is a diameter of the circle and its midpoint is the center of the circle.
 4. Set a compass at the midpoint of the line, with its drawing point at one of the endpoints, and draw the circle.
 5. To perform generic transformations, measure an angle of 2θ (positive counterclockwise) from the initially drawn diameter at the midpoint. A diameter drawn through the circle at this angle will intersect the circle at the points $(\sigma_{x1}, \tau_{xy1})$ and $(\sigma_{y1}, -\tau_{xy1})$. Values of the stresses may be found visually if the drawing is to scale, or through the use of trigonometry.
 6. To find the principal and maximum shear stresses, simply identify the right, left, and top/bottom points of the circle, visually (if drawn to scale) or trigonometrically.
-

Historically, Mohr's circle was a powerful computational tool to assist with calculations. With an accurate drawing (and appropriate use of compass, ruler, and protractor), it was relatively straightforward to determine the transformed stresses at any angle. However, since the 1980s handheld calculators have become quite common and the use of Mohr's circle as a computational tool has declined significantly. Recall that it is simply a graphical representation of the transformation equations; if we can quickly solve them directly, then there is no reason to consider the circle in the solution.

At this point Mohr's circle is, frankly, very inconvenient for solving plane stress problems. It does still have some value though. First, it is an easily understood visualization of stress transformations. Although the representation of stress in its components may change with coordinate transformations, the total state of stress does not; any point on a circle, regardless of its location, is the same distance from the center. In this sense it is also useful for cleanly representing different failure theories (beyond the scope of this text) and understanding their uses.

Perhaps the greatest value of Mohr's circle is in extending transformation concepts to certain three-dimensional problems. Clean equations such as (5.14) for transforming the stress or (5.22) for determining principal stresses do not exist in three dimensions. Rotation of the coordinate axes can occur in any plane, in any order. Analytical determination of the transformed state requires determination of eigenvectors and eigenvalues of the stress tensor.

However, we already have some experience with a 3D problem. If we consider plane stress (in the principal configuration) as a special case of triaxial stress, then everything completed so far with Mohr's circle must also hold for the third principal stress. In plane stress, the leftmost and rightmost points of Mohr's circle represent the in-plane principal stresses, while all other points on the circle appear when the element is rotated about the third axis (for example, the z -axis). The third normal stress, σ_z , does not change if the element is rotated about the z -axis. This implies that if we are to include it, it is not a part of the Mohr's circle we have already drawn! Instead, we must look at the two other planes and allow rotation about the x - and y -axes.

While this would seem prohibitively complicated at first, we already know all three principal stresses. We also know that in the xy -plane, those two principal stresses form the left and right boundaries of Mohr's circle. The same condition holds in the xz - and yz -planes. Thus we can draw three circles between σ_1 and σ_2 , σ_2 and σ_3 , and σ_1 and σ_3 . In plane stress, the last principal stress is always zero.

Figure 5.15 shows Mohr's circles drawn for the stress elements in Figure 5.13. In Figure 5.15(a), the in-plane principal stresses have opposite signs, so $\sigma_2 = 0$. In Figure 5.15(b), the in-plane principal stresses are both positive, so $\sigma_3 = 0$. This is a fine point, but has significant consequences. The maximum shear stress is the radius of Mohr's circle; in the 3D case there are three circles, therefore the radius of the *largest* circle is the maximum shear stress. When the in-plane principal stresses have opposite sign, the largest circle is the one drawn for that plane (Figure 5.15(a)), so the maximum shear stress matches equation (5.24). If the in-plane principal stresses have the same sign though, as in Figure 5.15(b), then equation (5.24) gives the radius of the smaller circle, which is not truly the maximum shear. The out-of-plane shear stress is greater.

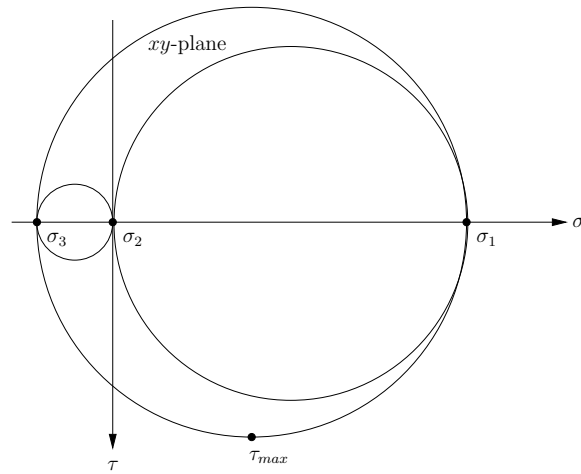
This is why the notation $\tau_{max,p}$ was used in equation (5.24), to indicate that it is not necessarily the absolute maximum shear stress. Based on Mohr's circle, equation (5.25) can be updated to always provide the maximum shear stress.

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \quad (5.34)$$

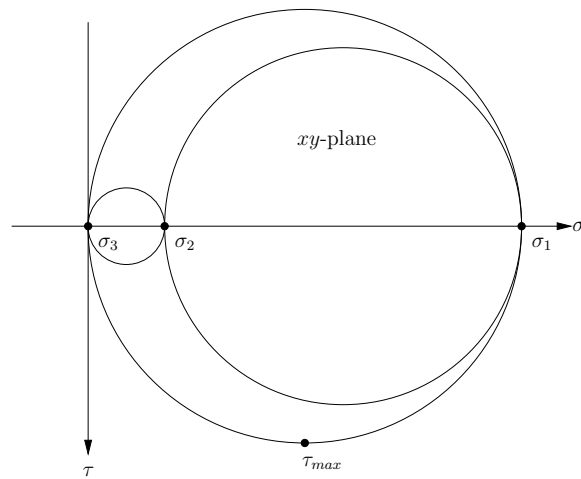
At this point it is no longer necessary to draw Mohr's circle for any plane stress problem. It is enough to use the transformation equations to solve for the new state of stress, or more often to solve for the in-plane principal stresses then include the zero out-of-plane stress to determine σ_1 , σ_2 , σ_3 , and τ_{max} .

5.4.1 General 3D Problems

Care must be taken with Mohr's circle for three-dimensional problems. One could interpret Figure 5.15 as allowing any 3D problem to be decomposed into a set of plane stress problems. Revisit Figures 5.7 and 5.8, as well as equations (5.11) – (5.13). All stresses lie in the xy -plane. Obviously z -stresses are oriented out-of-plane and would not affect the x - and y -force balances. This would seem to imply that any plane can be transformed through this process.



(a) In-plane principal stresses with opposite sign



(b) In-plane principal stresses with the same sign

Figure 5.15: Mohr's circle for triaxial stress

The problem with this conclusion is that shear stresses are always paired with perpendicular components; τ_{xz} may point in the z -direction, but its conjugate τ_{zx} is oriented in the x -direction. Therefore out-of-plane shear stresses interfere with the force balance of equation (5.11). Thus Mohr's circle and the plane stress transformations can only be extended to the triaxial case. General 3D problems are not manageable through this approach.

5.5 Thin-Walled Pressure Vessels

At this point we have developed a strong sense of what plane stress is. We understand the nature of the stresses and the importance of the coordinate system. We are capable of transforming the coordinates to find the stress components in any direction, including that which develops the greatest magnitude stresses. However, none of this is particularly valuable without a known initial state of stress. Fortunately, each of the surrogate models developed thus far are subsets of general plane stress: bars (σ_{xx}), shafts (τ_{xy}), and beams (σ_{xx} and τ_{xy}). The material in this section will introduce one additional model, pressure vessels.

5.5.1 Cylindrical Pressure Vessels

Consider a vessel filled with a fluid. Each particle of the fluid exerts a uniform force in all directions, known as **fluid pressure** (this effect is sometimes called **Pascal's law**). Analysis of the stress in this vessel is straightforward so long as some assumptions are valid.

1. The pressure vessel is a hollow, prismatic cylinder of uniform wall thickness
2. The pressure vessel is thin-walled; thickness t is much less radius of the cylinder R
3. The pressure p is uniform throughout the fluid

Assumption 1 is self-explanatory, but the other two warrant some discussion. If the vessel is considered thin-walled, then the force and stress are distributed through the thickness approximately uniformly and analysis may be reduced to the midplane (similar to analysis of bars reducing to the longitudinal axis). Structural models in which this holds are generally called **shells**. There is no rigorous law for what qualifies as “thin” but, similar to Saint-Venant's principle, there is a fairly standard rule-of-thumb; if $R > 10t$, then any approximation errors are negligible.

The final assumption requires some additional validation. Spatial variations in the pressure introduce some complicating factors, so we choose to neglect those. However, this cannot be done arbitrarily. The pressure in a fluid due to its own weight, the **hydrostatic pressure**, is given by:

$$p = \rho gh \tag{5.35}$$

in which ρ is the mass density of the fluid, g is gravitational acceleration, and h is the height of the fluid above the point in question. Logically this indicates that pressure is greatest at the bottom of the tank. For this reason, a municipal standpipe (Figure 5.16) would not satisfy the third assumption. The weight of dozens of meters of water would make the pressure at the bottom of the pipe significantly greater than at the top.



Figure 5.16: Standpipe water tower

Even a horizontally-oriented tank could experience significant variation from top to bottom. It is for this reason that all vessels analyzed through this approach are externally pressurized. Boilers filled with steam, air compressors, and natural gas canisters are all examples in which the applied pressure is much greater than the hydrostatic pressure; in these cases, p is the pressure to which the vessel has been pressurized, and variations due to height are considered negligible.

One final point of clarification with regards to the pressure is important. The fluid inside the pressure vessel pushes outward against it. External to the pressure vessel, the atmosphere applies a pressure field inward. The pressure p , as used here, is the *difference* between these two pressures. It is what is called a **gage pressure**; consider it a net pressure that allows us to exclude atmospheric pressure from our analyses.

With clever use of free-body diagrams, pressure vessels may be analyzed using static equilibrium alone. Consider the diagram shown in Figure 5.17. In this diagram, a section has been taken along the central plane of the cylinder. The section *includes the fluid*. Therefore internal forces on the fluid, the pressure, are included in the free-body diagram and forces between the fluid and the curved vessel wall are excluded.

The forces on this section consist of the fluid pressure, pushing normal to the fluid, and the reaction from the walls of the pressure vessel, also normal to the section. Notice that there are no radial forces in this section because the fluid was included as a part of the object. The internal force of the cylinder F_c is limited to the sides of the cylinder and largely excluded from the endcaps; from experience in Chapter 2, we understand that most of an applied force is

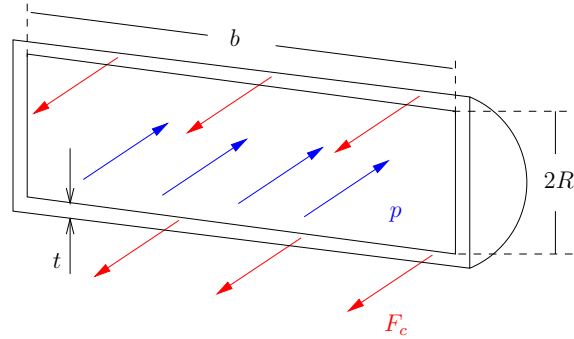


Figure 5.17: Longitudinal section of the pressure vessel

borne by the closest supports. In this case, the walls of the cylinder bear most of the pressure so neglecting loads in the endcaps is fair. Also notice that R measures the *interior* radius of the vessel. The internal length of the tank (i.e., the length of the fluid neglecting the solid endcaps of the cylinder) is b . Because the pressure is uniform and the walls are thin, it is reasonable to assume that F_c is also uniformly distributed. The total force due to the pressure is given by:

$$F_p = pA = p(2Rb) \quad (5.36)$$

Applying static equilibrium:

$$\begin{aligned} F_p &= F_c \\ 2pRb &= \sigma_H A = \sigma_H(2bt) \\ \sigma_H &= \frac{pR}{t} \end{aligned} \quad (5.37)$$

Thus the stress around the walls of the pressure vessel is given by (5.37). σ_H is called the **hoop stress** or **circumferential stress**. The name hoop stress is easily understood when practical examples are considered. Barrels, drums, and the like are very often banded by sturdy hoops to provide additional stiffness against the load of their contents.

Another section, parallel to the endcaps (Figure 5.18), reveals internal forces perpendicular to the previous case. Once again, the fluid is included as part of this free-body diagram. Following an identical process to that used to determine the hoop stress, another stress component can be determined.

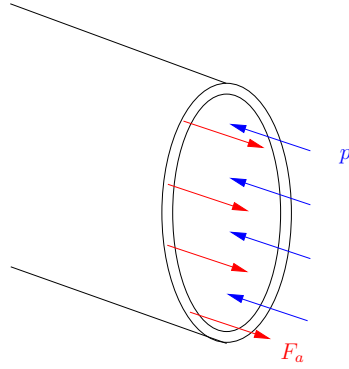


Figure 5.18: Circumferential section of the pressure vessel

$$F_p = p(\pi R^2) \quad (5.38)$$

$$\begin{aligned} F_p &= F_a \\ \pi R^2 p &= \sigma_A A = \sigma_A (2\pi R t) \\ \sigma_A &= \frac{pR}{2t} \end{aligned} \quad (5.39)$$

σ_A is known as the **axial stress** or **longitudinal stress** and is oriented along the length of the cylinder. In simplest terms, it is the material response to the force of the fluid pushing on the endcaps. The axial stress is half of the hoop stress.

Now consider the outer surface of the pressure vessel. There is no force pushing radially on it in either Figure 5.17 or 5.18 (due to our use of gage pressure). This implies that there is no radial component of the stress. Furthermore, there is no shear component in either of the free-body diagrams.

On the interior surface of the pressure vessel, the p is oriented directly against the cylinder. Thus, the radial stress $\sigma_r = p$. This is problematic because it would prevent us from modeling the pressure vessel using plane stress. Fortunately, our restriction to thin-walled vessels provides some relief. Because the walls are thin, the radial stress must very quickly vanish through the thickness to reach the zero value on the outer surface. Additionally, and perhaps more significantly, the geometric limit defined for a wall to be thin, $R > 10t$ leads to the hoop stress being at least ten times greater than the radial stress.

$$\frac{pR}{t} = \frac{p(10)}{(1)} \quad (5.40)$$

Thus, neglecting the radial stress and considering the state of stress throughout the thickness to consist only of the hoop and axial stress is a reasonable approximation.

Since the hoop and axial stresses are the only two stresses and both are normal stresses, they must be the in-plane principal stresses. The in-plane

maximum shear stress is defined by their difference.

$$\begin{aligned}\sigma_{1p} &= \sigma_H \\ \sigma_{2p} &= \sigma_A \\ \tau_{max,p} &= \frac{\sigma_H - \sigma_A}{2} = \frac{pR}{4t}\end{aligned}\tag{5.41}$$

The third stress component is zero, leading to the following triaxial principal stresses and absolute maximum (out-of-plane) shear stress.

$$\begin{aligned}\sigma_1 &= \sigma_H \\ \sigma_2 &= \sigma_A \\ \sigma_3 &= 0 \\ \tau_{max} &= \frac{\sigma_1 - \sigma_3}{2} = \frac{pR}{2t}\end{aligned}\tag{5.42}$$

5.5.2 Spherical Pressure Vessels

Spherical pressure vessels, while perhaps less common than cylindrical, appear often enough to discuss here. One interesting feature of a sphere is its uniformity; on the surface of a sphere, direction is essentially meaningless. The magnitude of the gradient is independent of orientation. Because of this, any section taken as a **great circle** of the sphere (a circle through the diameter of the sphere) will yield the same results. The details of the derivation are left as an exercise, but the result for a thin-walled spherical pressure vessel is:

$$\sigma_M = \frac{pr}{2t}\tag{5.43}$$

σ_M denotes the **membrane stress**, and is the same in every in-plane direction. Following the discussion for cylindrical pressure vessels, the radial stress is neglected, leaving the following principal and maximum shear stresses.

$$\begin{aligned}\sigma_1 &= \sigma_M \\ \sigma_2 &= \sigma_M \\ \sigma_3 &= 0 \\ \tau_{max} &= \frac{\sigma_1 - \sigma_3}{2} = \frac{pR}{4t}\end{aligned}\tag{5.44}$$

Example 5.3 (Pressure vessels).

PROBLEM STATEMENT:

Determine the strain on the surface of an air cylinder used for scuba diving.

KNOWNs:

- The manufacturer has provided the following dimensions for the air tank:
 - The outer diameter is 7.25 in.
 - The inner diameter is 6.39 in.
 - The length of the tank is 26 in.
- The tank is made of aluminum, so from Appendix B:
 - $E = 10.1 \times 10^6$ psi
 - $\nu = 0.33$
- When filled, the tank is pressurized to 3000 psi

ASSUMPTIONs:

- The tank experiences only small deformation
- The tank is completely cylindrical
- End effects and stress concentrations from the valve and fittings are negligible
- The wall of the tank is thin

ANALYSIS:

With these assumptions, this problem meets the requirements for analysis as a thin-walled cylindrical pressure vessel. To use equations (5.37) and (5.39), the internal radius and thickness are required.

$$R = \frac{D}{2} = \frac{6.39}{2} = 3.195 \text{ in.} \quad (5.45)$$

$$t = R_o - R = \frac{7.25}{2} - 3.195 = 0.43 \text{ in.} \quad (5.46)$$

These values may be substituted directly into equations (5.37) and (5.39).

$$\sigma_H = \frac{(3000 \text{ psi})(3.195 \text{ in.})}{0.43 \text{ in.}} = 22,300 \text{ psi} \quad (5.47)$$

$$\sigma_A = \frac{\sigma_H}{2} = 11,150 \text{ psi} \quad (5.48)$$

There are no shear stresses in this coordinate system.

The stresses are now fully defined. The stress is related to the strain through Hooke's law. Substituting the stresses into Hooke's law for plane stress, equations (5.5):

$$\begin{aligned} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} &= \frac{1}{10.1 \times 10^6} \begin{bmatrix} 1 & -0.33 & 0 \\ -0.33 & 1 & 0 \\ 0 & 0 & \frac{1-0.33}{2} \end{bmatrix} \begin{Bmatrix} 22,300 \\ 11,150 \\ 0 \end{Bmatrix} \\ &= \begin{Bmatrix} 0.00289 \\ 0.00206 \\ 0 \end{Bmatrix} \end{aligned} \quad (5.49)$$

DISCUSSION:

- The length of the cylinder does not impact the stress experienced by the pressure vessel. Only the radius and thickness are relevant.
- In this case the radius of the pressure vessel is about 7.5 times the thickness of the wall. This may result in some error in the solution, but the assumption of a thin-walled pressure vessel is not wildly inappropriate.
- The calculated strains are quite low, supporting the small deformation assumption.
- In general, as in this example, modeling of thin-walled pressure vessels is quite direct, with little effort required.

5.6 Combined Loading

Our toolbox at now consists of four primary surrogate models:

1. Bars
2. Shafts
3. Beams
4. Pressure vessels

Each of these include numerous additional details and features such as strain energy, stress and deformation, static indeterminacy, etc. Each is useful in its own right. However, practical structures very commonly do not fit perfectly into the requirements for any one of these models. Is the only recourse then to complete a detailed three-dimensional analysis?

One point we have noticed throughout our work is that stress and deformation are linearly related to the internal loads. The linearity of these relationships is key to broadening their use. Recall that the principal of superposition holds

for all linear transformations, not just subsets. With that in mind, as long as we can decompose a structure into pieces that satisfy the requirements of the individual surrogate models, then it is possible to use multiple models in a single **combined loading** problem. It has already been discussed in Section 2.6, but it is worth emphasizing again that several conditions are critical for superposition to be admissible:

1. The material experiences only small deformation
This condition is necessary for two reasons. First, following discussion in Section 1.4, the strain-displacement (kinematic) relations become nonlinear if deformation gradients are large. This nonlinearity interferes with superposition. Second, large displacements can change the geometry of the problem and affect how different loads interact with each other, as shown in Figure 2.18. How large is too large is a judgment to be made by the analyst. For example, large deformation in the context of shaft models is defined by the rate of twist $\frac{d\phi}{dx}$. A sufficiently long shaft could have an angle of twist $\phi = 90^\circ$ at the end of the member, but still be considered small deformation for the purposes of shaft theory. However, if there is another structure or load at that end of the shaft, it will be completely changed by the deformation of the shaft, preventing the use of superposition.
2. The material must be linearly elastic
Linearly elasticity is a constitutive theory connecting stress (load) to strain (deformation) through the constitution of the matter. Only Hooke's law allows for a linear relationship between stress and strain. If a material is such that Hooke's law does not adequately describe it (many polymers are good examples of this), then superposition is inappropriate.
3. The structure and loading are quasistatic
This requirement may seem obvious given its use in deriving every single surrogate model used in this text, but there is at least one case that deserves additional notice. As explained in Section 3.4, shafts rotating at a constant velocity may be modeled in the same manner as truly static shafts. However this is not the case if the shaft is not in isolation. For example if there is a transverse force acting on it, then as the shaft rotates it is constantly deflecting in a new direction. There are acceleration and inertial forces that cannot be neglected.

Fortunately, none of the above requirements are new. All of our surrogate models include these three conditions as fundamental assumptions necessary to their function.

5.6.1 Stress under Combined Loading

The use of superposition for combined loading problems is a direct extension of the work completed throughout this text with clear steps to be followed.

1. Determine all loads and reactions on the entire structure
2. Deconstruct the geometry
3. Decompose the loading
4. Apply surrogate models
5. Combine stress/deformation results

5.6.1.1 Determination of Reactions

The first step in a combined loading problem is nothing more than application of static equilibrium to the entire structure. This should be second nature at this point, but for clarity begin with a free-body diagram of the entire structure. In many cases, the structure is three-dimensional, so the equations of equilibrium should reflect that. Moments in particular should be defined through the use of equation (1.14) to avoid errors.

$$\begin{aligned}\sum \mathbf{F} &= 0 \\ \sum \mathbf{M} &= \sum (\mathbf{r} \times \mathbf{F}) = 0\end{aligned}\tag{5.50}$$

5.6.1.2 Deconstruction of Geometry

All of the surrogate models discussed in this text require that structures meet certain shape requirements; long and slender, straight, cylindrical, circular, etc. Relatively few real structures do. In order to use these models, a problem must be broken into appropriate pieces. This deconstruction is performed through careful use of sections. Naturally, taking a section results in internal forces that must then be resolved through static equilibrium.

Consider the bracket shown in Figure 5.19(a). As it is not straight, it does not meet the requirements for any model discussed here. However, when a section is taken at point A in Figures 5.19(b) and 5.19(c), then the two sides of the section are both long, slender, straight, and suitable for use in bar, shaft, or beam theory as appropriate.

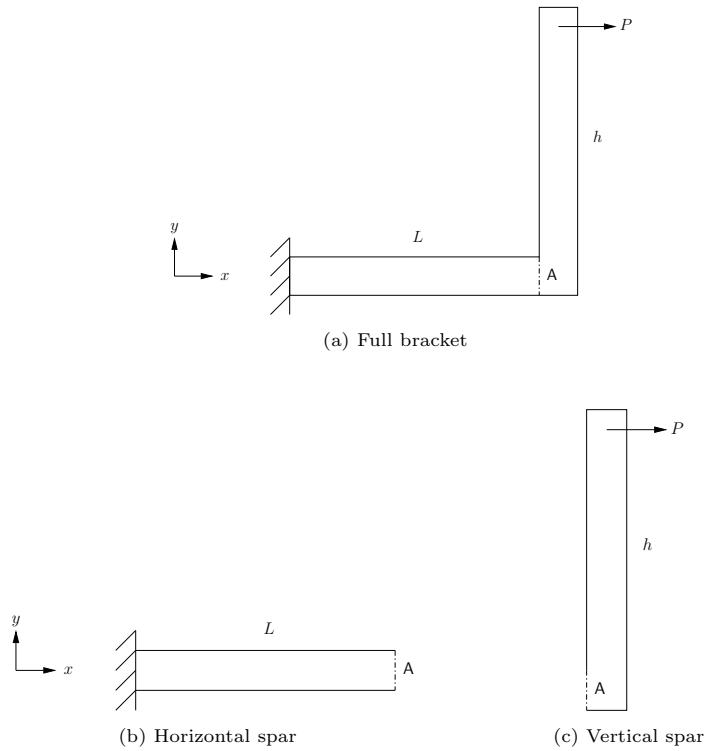


Figure 5.19: Deconstruction of an angle bracket

5.6.1.3 Decomposition of Loads

The primary difference between each surrogate model is the type of loading applied. It is possible that a combined loading problem requires no deconstruction of the geometry at all, only separation of load types to apply different models. For example, a pressurized cylindrical tank supporting its own weight on two feet, as shown in Figure 5.20, is directly identified as both a pressure vessel and a beam problem.

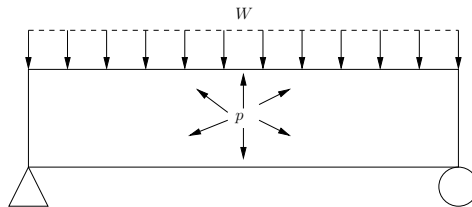


Figure 5.20: Beam and pressure vessel

The structure in Figure 5.20 did not require any decomposition of the load, only recognition of appropriate models to apply. A second example in Figure

5.21 shows a straight, slender member loaded by a force acting at an angle to the member. This is neither a bar, which requires axial forces, nor a beam, which requires perpendicular forces to create moments. However, by simply decomposing the force into x - and y -components, the problem clearly reduces to a combination of bar and beam.

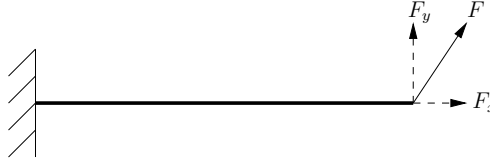


Figure 5.21: Bar and beam

In a more complex example, it is sometimes possible to avoid geometric deconstruction through careful application of load decomposition. Consider the bracket in Figure 5.19(a). Suppose we do not particularly care about the stress in the upright segment, to the right of A. In that case, it is not necessary to fully understand the forces acting on that portion of the bracket and instead the force may be moved to point A to act on the horizontal spar directly. However, the force was originally acting at a distance from point A, so for the load to remain unchanged when the force moves, a moment equal to the force times the distance moved, $M = Ph$, must be included, as shown in Figure 5.22.

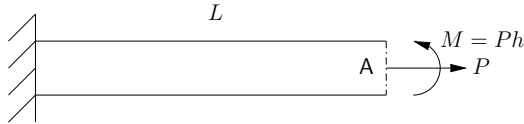


Figure 5.22: Statically equivalent loading

The horizontal spar in Figure 5.22 is now clearly a combination of bar and beam. This process is known as **statically equivalent loading**. Another example is the conversion distributed loads to point loads; statically equivalent loading is the alteration of an applied load in such a way that the reaction forces and moments do not change to maintain static equilibrium.

Care must be taken with statically equivalent loading. Recall from Chapters 2 – 4 that a distributed load may be converted to a point load to simplify the statics problem, but must remain a distributed load for stress and strain analysis. This is because the true location and distribution of the load directly impact the internal forces and displacements of every point. Statically equivalent loads do not affect the reaction forces, but will alter everything between their original state and their equivalent state. For this reason, statically equivalent loading should only be used when specific information is necessary. In Figure 5.22, all information about the stress and deformation of the upright segment of the bracket was lost when the equivalent load was applied. This

approach is only useful if, for example, stress at the left end of the bracket was the only information required.

5.6.1.4 Application of Surrogate Models

This step is perhaps the most straightforward part of the process for solving combined loading problems. Once geometry and loads have been fully decomposed into their constituents, simply apply the appropriate model. Ensure that the assumptions and restrictions for each model are satisfied, then solve each problem individually for stress and/or deformation as necessary.

Keep in mind some of the more restrictive assumptions for each model. For example, shafts require circular cross-sections, beams require cross-sections that are symmetric about the y -axis, and pressure vessels require thin walls. A structure that satisfies the assumptions of one model may not be suitable for another, even if the loading is appropriate.

Take care to solve each sub-problem using the same coordinate system. This is critical for being able to superposition the results. Consider the bracket in Figures 5.19(b) and 5.19(c). If the x -axis is parallel to the horizontal spar of Figure 5.19(b), then the bending stress in that segment is σ_{xx} . The upright spar in Figure 5.19(c) is also bending, but in this case the y -axis is parallel to the neutral surface, so the bending stress would be σ_{yy} .

5.6.1.5 Superposition of Results

With the stresses from each model individually determined, it is possible to combine them into a complete representation of the stress field. Care should be taken though. Only the same components of stress may be superpositioned; σ_{xx} cannot be added to σ_{yy} or τ_{xy} . Location is also critical. Each of the stresses may be a function of x , y , and z . We cannot simply take the greatest value of each and add them.

At this point it is useful, and in fact critical, to consider stress elements. At any location where one or more stresses appear to peak, a plane stress element should be drawn. Each component of the stress from each sub-problem is to be included on the element and summed as appropriate. From the element, principal and maximum shear stresses *for that location* can be determined. These values can then be compared at the different elements to determine the worst case for the structure and ultimately to predict failure.

Note that the steps outlined in this section are applicable to all problems, but do not necessarily need to be applied to every single problem or applied in the order listed. For example, if the state of stress is to be determined at a defined location, then detailed deconstruction of geometry may be unnecessary. In that case it may also be beneficial to determine support reactions after loads have been converted to statically equivalent forms. Other possible combinations exist that may be more efficient for specific problems. The five steps listed here are always applicable though.

Example 5.4 (Combined loading).

PROBLEM STATEMENT:

Determine the maximum xy -stresses and the associated principal and maximum shear stresses on the tank of liquified natural gas shown in Figure 5.23.

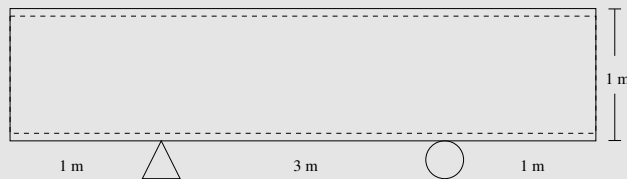


Figure 5.23: Holding tank for liquified natural gas (LNG)

KNOWNs:

- $L = 5$ m
- $D = 1$ m
- $t = 0.5$ cm
- Gas is pressurized to $p = 1.5$ MPa
- Total weight of tank and gas: $W = 30$ kN

ASSUMPTIONs:

- Weight is approximately uniformly distributed: $w(x) = \frac{W}{L} = 6000$ N/m
- The tank experiences only small deformation
- The tank and its contents are static
- Hydrostatic variations in pressure are negligible
- End effects may be neglected
- Tank material is linearly elastic, continuous, homogeneous, and isotropic

ANALYSIS:

The loading for this problem consists of internal pressure and weight force transverse to the longitudinal axis. Therefore this gas tank is most likely a combination of pressure vessel and beam models. Following the steps described

in Section 5.6.1, we begin with a free-body diagram of the entire structure to determine reactions.

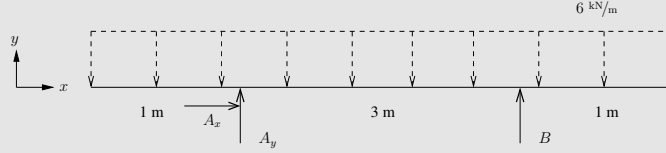


Figure 5.24: Free-body diagram for Example 5.4

Note that the pressure inside the tank does not appear in the free-body diagram of Figure 5.24. Simple observation shows that $A_y = B$ due to symmetry. Apply static equilibrium to solve for the reaction forces A_x , A_y , and B .

$$\sum F_x = 0 = A_x \quad (5.51)$$

$$\sum F_y = 0 = A_y + B - \int_0^5 w(x) dx \quad (5.52)$$

$$= 2A_y - (6000)(5) \implies A_y = B = 15,000 \text{ N}$$

In this example there is no deconstruction of geometry necessary since the tank is already one-dimensional. Progressing to the next step, decomposition of loads leads to two problems: one in which the pressure acts alone and another in which the weight force acts alone. These two load cases lead us directly into the appropriate surrogate models.

Cylindrical pressure vessel:

The internal pressure field is a clear indication that a pressure vessel model should be considered. Before proceeding with analysis, however, it is critical that we confirm that the assumptions and restrictions of the model are satisfied.

- Thin-walled $\implies t < \frac{R}{10}$
 - Recall that R is the *interior* radius
 - $t = 0.005 \text{ m} < \frac{1}{10} \left(\frac{D}{2} - t \right) = 0.0495 \text{ m}$
 - This condition is satisfied
- The tank is cylindrical
- Pressure is assumed to be uniform

Thus modeling this part of the tank as a cylindrical, thin-walled pressure vessel is acceptable. Applying equations (5.37) and (5.39):

$$\sigma_H = \frac{pR}{t} = \frac{(1.5 \text{ MPa})(0.495 \text{ m})}{0.005 \text{ m}} = 148.5 \text{ MPa} \quad (5.53)$$

$$\sigma_A = \frac{pR}{2t} = \frac{\sigma_H}{2} = 74.25 \text{ MPa} \quad (5.54)$$

These stresses hold at every location on the surface of the cylinder (away from endcaps). The axial stress is oriented in the x -direction.

Beam:

The transverse load due to the weight of the tank will cause bending, leading us to consider a beam model. Once more, this choice must be validated.

- The cross-section is circular, therefore symmetric
- The member is relatively long and slender, and is straight
- We assume shear strain is negligible

With this, beam analysis is an appropriate choice. To determine the stress, the internal moment is necessary. Begin with shear and moment diagrams.

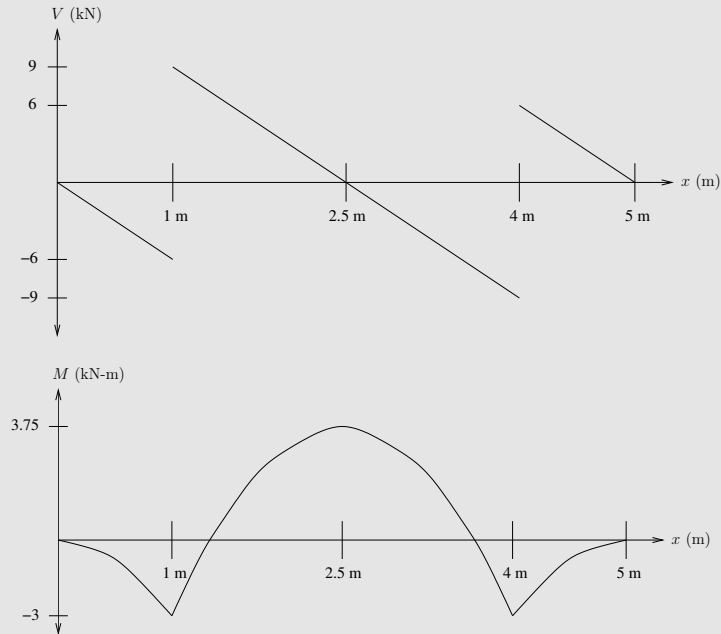


Figure 5.25: Shear and moment diagrams for Example 5.4

The maximum shear force is $V = 9$ kN at $x = 1$ m and the maximum moment is $M = 3.75$ kN-m at $x = 2.5$ m. To determine stress, the second area moment I is also required.

$$I = \frac{\pi}{64} (D^4 - (D - 2t)^4) = 0.00193 \text{ m}^4 \quad (5.55)$$

For shear stress, the first area moment Q must be found as well. Since shear stress peaks at the neutral axis, Q is determined for a semicircular arc. From Appendix A:

$$Q = 2 \left(R + \frac{t}{2} \right)^2 t = 0.00248 \text{ m}^3 \quad (5.56)$$

We now compute the maximum stresses. Keep in mind that these values occur at two different locations. The bending stress achieves its maximum value at $x = 2.5 \text{ m}$ at the maximum distance from the neutral axis, $y = \pm 0.5 \text{ m}$.

$$\sigma_{xx}(2.5, \pm 0.5) = -\frac{My}{I} = -\frac{(3.75 \text{ kN}\cdot\text{m})(\pm 0.5 \text{ m})}{0.00193 \text{ m}^4} = \mp 0.97 \text{ MPa} \quad (5.57)$$

These maximum stresses have equal magnitude but opposite direction. When evaluating a beam in isolation, it is often sufficient to consider only the magnitude. However, in combined loading problems sign can make critical differences in the superposition process, so all stresses should be considered and evaluated. In this case, these stresses will be combined with those of a pressure vessel, which are tensile. Thus, the most significant stresses will develop when the bending stress is also tensile. For that reason, the maximum bending stress considered in this problem is the positive stress on the *bottom* of the tank. The shear stress at the top and bottom surfaces of a beam are zero.

The maximum shear stress in the beam occurs at $x = 1 \text{ m}$ on the neutral axis, where the bending stress is zero.

$$\tau_{xy}(1, 0) = -\frac{VQ}{Ib} = -\frac{(9 \text{ kN})(0.00248 \text{ m}^3)}{(0.00193 \text{ m}^4)(2 \times 0.005 \text{ m})} = -1.16 \text{ MPa} \quad (5.58)$$

With the stresses from all surrogate models fully-defined, we proceed to superposition. Stress elements must be drawn at each location where the stresses peak, with appropriate coordinates noted and all relevant stresses included. Figure 5.26(a) shows a stress element taken on the bottom of the tank at the point $(x, y) = (2.5, -0.5)$. This element is parallel to the xz -plane. If one were to look at the tank from underneath, the x -axis would be oriented to the right and the z -axis would point upward. In this element, the hoop stress becomes a z -stress. The axial pressure vessel stress and bending stress are both normal x -stresses (shown individually on the left face of the element), so may sum (represented on the right face).

The stress element of Figure 5.26(b) is located at $(x, y) = (1, 0)$ on the front of the tank. It is parallel to the xy -plane, thus its axes are identical to those in the free-body diagram of Figure 5.24. In this case, the hoop stress becomes a y -stress, while the axial and shear stresses maintain their definitions. None of these stresses consist of the same components, so superposition does not require summation, only placement on the same stress element.

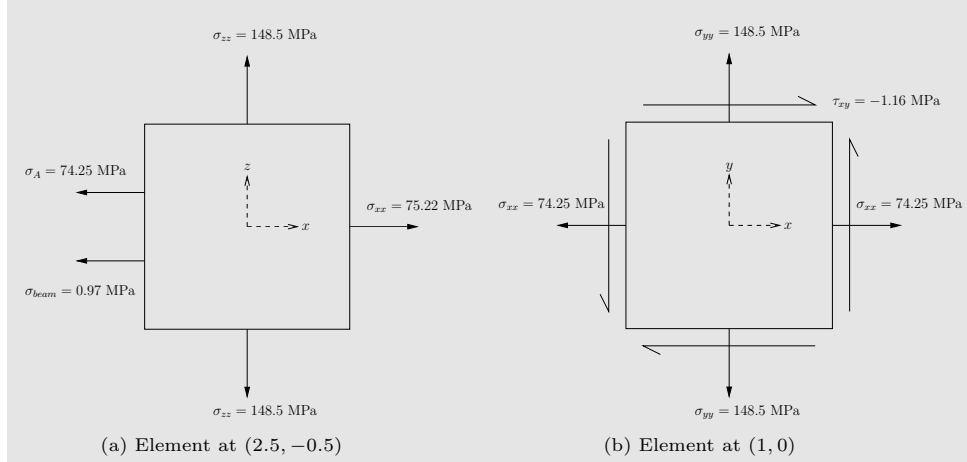


Figure 5.26: Stress elements for Example 5.4

The stress element of Figure 5.26(a) contains no shear stresses, so is already in the principal configuration. The principal and maximum shear stresses are:

$$\begin{aligned}
 \sigma_1 &= 148.5 \text{ MPa} \\
 \sigma_2 &= 75.22 \text{ MPa} \\
 \sigma_3 &= 0 \text{ MPa} \\
 \tau_{max} &= \frac{\sigma_1 - \sigma_3}{2} = 74.25 \text{ MPa}
 \end{aligned} \tag{5.59}$$

The element of Figure 5.26(b) is not in the principal configuration already, so in-plane principal stresses must first be calculated.

$$\begin{aligned}
 \sigma_{1p} &= \frac{148.5 + 74.25}{2} + \sqrt{\left(\frac{148.5 - 74.25}{2}\right)^2 + (-1.16)^2} = 148.52 \text{ MPa} \\
 \sigma_{2p} &= \frac{148.5 + 74.25}{2} - \sqrt{\left(\frac{148.5 - 74.25}{2}\right)^2 + (-1.16)^2} = 74.23 \text{ MPa}
 \end{aligned} \tag{5.60}$$

The triaxial principal and maximum shear stresses are:

$$\begin{aligned}
 \sigma_1 &= 148.52 \text{ MPa} \\
 \sigma_2 &= 74.23 \text{ MPa} \\
 \sigma_3 &= 0 \text{ MPa} \\
 \tau_{max} &= \frac{\sigma_1 - \sigma_3}{2} = 74.26 \text{ MPa}
 \end{aligned} \tag{5.61}$$

Thus in this case the two stress elements are nearly identical.

DISCUSSION:

- Stress elements should be considered at every point of concern in the structure, typically where stresses in the cardinal directions are highest.
- The calculated stresses are used in appropriate failure theories to determine if the structure will yield.
- Recall from Chapter 4 discussion that bending effects often dominate the stress of the structure. This is not the case for this example for two reasons.
 - The only transverse load is the weight of the holding tank. As mentioned throughout the text, weight forces are very often significantly lower than the loads a structure is designed to support, and so produce negligible stresses. Notice that both the bending and shear stresses cause almost no change at all in the principal and maximum shear stresses compared to the pressure vessel acting alone.
 - The supports are located in such a way as to nearly minimize the moment due to the weight of the tank. This is a choice. By designing not only the structure but also the way the structure is supported, an engineer can very effectively manage loads and stresses.
- At this point it would be prudent to check that deformation is satisfactorily small. Without calculating all strains and displacements, we can convince ourselves this is the case.
 - The beam stresses are quite small, indicating minimal curvature of the tank.
 - The pressure vessel stresses are uniform and will result primarily in a change of volume rather than change of shape.
 - The maximum stresses are well below the yield stress of the tank material (supposing it is steel). Metals typically yield before significant deformation occurs.
 - These arguments are satisfactory, but remember that the only way to be sure deformation is small enough is to compute it and check.

With practice, it becomes easy to compress the solution steps. In the previous example, one could observe without completing any work that the tank behaved as a beam due to the weight load, a pressure vessel due to the internal fluid, required no geometric or load decomposition, and that the peak stress is very likely on the bottom surface of the tank near the center where bending stress and axial pressure-vessel stress align. Further examples will develop this skill.

Example 5.5 (Combined loading).

PROBLEM STATEMENT:

You are trying to disassemble some furniture before you move out of your apartment. The screws holding the pieces together are stuck, so the allen wrench you are using does not move. Determine the principal and maximum shear stresses acting on the stress element at point A on the allen wrench (also known as a hex key) in Figure 5.27.

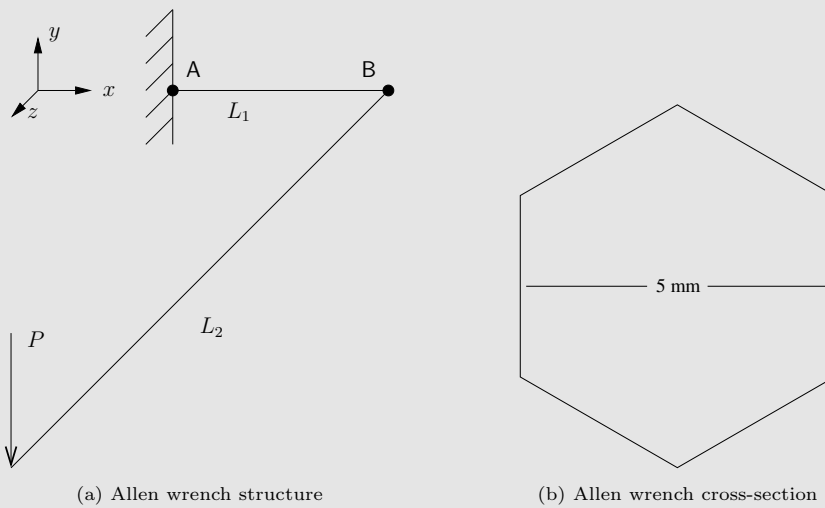


Figure 5.27: Schematic for Example 5.5

KNOWNs:

- $L_1 = 35$ mm
- $L_2 = 80$ mm
- $D = 5$ mm
- $P = 40$ N
- Axes are oriented as shown in Figure 5.27(a), with A at the origin

ASSUMPTIONs:

- Material is isotropic, homogeneous, continuous, and linearly elastic
- Material experiences small deformation

- End effects and stress concentrations are negligible
- Force P is insufficient to loosen the screw (i.e., the wrench is static)
- The wrench is fully engaged with the screw such that it behaves as a fixed boundary

ANALYSIS:

Since the stress is wanted only at point A, it is not necessary to determine maximum xyz -stresses for the entire structure. Additionally, the steps described in Section 5.6.1 may be performed out of order to streamline the process. Begin by deconstructing the geometry to remove the long arm L_2 and reduce the analysis to a single straight member L_1 extending from point A to point B.

The force P acting on the end of the long arm is replaced by a statically equivalent load acting at B. Since P operates at a distance from the elbow, a moment must be generated.

$$(P_y)(L_z) = M_x \quad \implies \quad -(-40)(0.08) = 3.2 \text{ N-m} = M_x \quad (5.62)$$

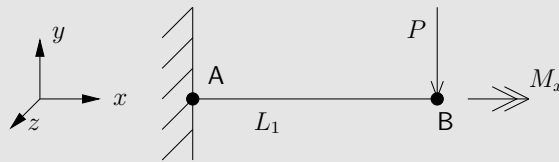


Figure 5.28: Statically equivalent loading on reduced geometry

At this point it is clear that the member is subjected to both bending and torsion loads, so should be modeled as a beam and shaft.

Shaft:

As always, begin by confirming that shaft model applies:

- Member is long and slender
 - L_1 is 35 mm, which is significantly greater than $D = 5$ mm
- Member is loaded by an axial torque
- The cross-section is hexagonal
 - Shaft theory requires a circular cross-section

It is not possible to proceed with solving this problem using the surrogate models currently available.

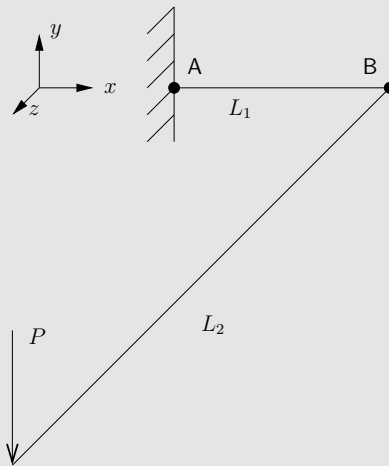
Example 5.6 (Combined loading).

PROBLEM STATEMENT:

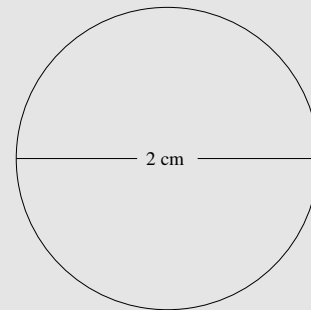
Before leaving for vacation, you notice one of your tires is flat and you (correctly) decide to change it. Before jacking the car up, you use a tire iron to loosen the lugnuts, but they do not give. Determine the principal and maximum shear stresses acting on the stress element at point A on the tire iron (also known as a lugwrench) in Figure 5.29.



(a) A commercially available tire iron



(b) Tire iron schematic



(c) Tire iron cross-section

Figure 5.29: Schematic for Example 5.6

KNOWNs:

- $L_1 = 0.2$ m
- $L_2 = 0.75$ m
- $D = 2$ cm
- $P = 250$ N
- Axes are oriented as shown in Figure 5.29(b), with A at the origin

ASSUMPTIONs:

- Material is isotropic, homogeneous, continuous, and linearly elastic
- Material experiences small deformation
- End effects and stress concentrations are negligible
- The wrench is solid with constant diameter; the flared end at the lugnut is neglected
- Force P is insufficient to loosen the lugnut (i.e., the wrench is static)
- The wrench is fully engaged with the lugnut such that it behaves as a fixed boundary

ANALYSIS:

Note that schematically this problem is identical to Example 5.5, with the exception of the cross-section shape. The dimensions change, but Figure 5.29(b) exactly matches Figure 5.27(a). Therefore all discussion of the problem prior to solution is identical to that in Example 5.5 and not repeated here.

We begin by reducing the problem to the segment AB and generating statically equivalent loads at point B.

$$M_{Bx} = -(P_y)(L_z) = -(-250)(0.75) = 187.5 \text{ N}\cdot\text{m} \quad (5.63)$$

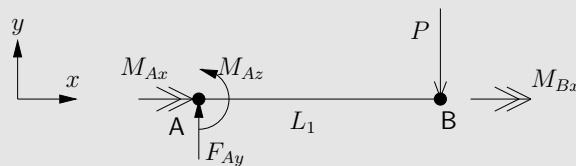


Figure 5.30: Statically equivalent loading on reduced geometry

At this point it is clear that the member is subjected to both bending and torsion loads, so should be modeled as a beam and shaft. Before proceeding with that analysis, first solve for the reactions shown in Figure 5.30. Recall that the fixed boundary provides all necessary reactions, in this case a y -force and x - and z -moments.

$$\sum F_y = 0 = F_{Ay} - P \implies F_{Ay} = 250 \text{ N} \quad (5.64)$$

$$\sum M_{OA,x} = 0 = M_{Ax} + M_{Bx} \implies M_{Ax} = -187.5 \text{ N}\cdot\text{m} \quad (5.65)$$

$$\sum M_{OA,z} = 0 = M_{Az} - PL_1 \implies M_{Az} = (250 \text{ N})(0.2 \text{ m}) = 50 \text{ N}\cdot\text{m} \quad (5.66)$$

Shaft:

As always, begin by confirming that shaft model applies:

- Member is long and slender
 - L_1 is 0.2 m, which is significantly greater than $D = 0.02 \text{ m}$
- Member is loaded by an axial torque
- The cross-section is circular

The stress on the surface of a shaft is given by:

$$\tau = \frac{TR}{J} \quad (5.67)$$

where τ is the shear stress parallel to the longitudinal axis, T is the internal torque, R is the shaft radius, and J is the polar second area moment. Since there are only couple torques applied at either end of the shaft, the internal torque must match them for the entire length and sections are unnecessary.

$$\begin{aligned} T &= M_{Bx} = 187.5 \text{ N}\cdot\text{m} \\ R &= \frac{D}{2} = 0.01 \text{ m} \\ J &= \frac{\pi D^4}{32} = 1.571 \times 10^{-8} \text{ m}^4 \\ \tau &= 119.4 \text{ MPa} \end{aligned} \quad (5.68)$$

This shear stress is uniform across the surface of the shaft, oriented longitudinally and circumferentially.

Beam:

As always, begin by confirming that beam model applies:

- Member is long and slender
 - L_1 is 0.2 m, which is significantly greater than $D = 0.02$ m
- Member is loaded by a bending moment
- The cross-section is circular, therefore symmetric
- Shear strain is neglected

Begin with shear and moment diagrams.



Figure 5.31: Shear and moment diagrams for Example 5.6

The bending stress on a beam is given by:

$$\sigma = -\frac{My}{I} \quad (5.69)$$

where σ is the normal stress parallel to the longitudinal axis, M is the internal bending moment, y is the distance from the neutral axis, and I is the second area moment. To maximize the bending stress, consider the greatest magnitude of the moment shown in Figure 5.31 and the greatest distance from the neutral surface.

$$\begin{aligned} M &= -50 \text{ N-m} \\ y &= \frac{D}{2} = 0.01 \text{ m} \\ I &= \frac{\pi D^4}{64} = 7.854 \times 10^{-9} \text{ m}^4 \\ \sigma &= 63.7 \text{ MPa} \end{aligned} \quad (5.70)$$

This normal stress is oriented longitudinally (i.e., it is σ_{xx}) and occurs at the top surface of the left edge of the beam, $(x, y) = (0, 0.01)$.

Non-uniform bending of beams also results in transverse shear stress.

$$\tau = -\frac{VQ}{Ib} \quad (5.71)$$

where τ is the transverse shear stress, V is the shear force, I is the second area moment of the cross-section, b is the material width at the location where the stress is calculated, and Q is the first area moment. The shear stress can only be calculated at a location where the sides are parallel to the y -axis, so the neutral surface. This is also the location of maximum shear stress. Since A is located at the top of the structure, where shear stress is 0, it is not necessary to calculate.

For the sake of illustration, we will still determine the shear stress τ . Q may be found using Appendix A. To maximize the shear stress, consider the greatest magnitude of the shear force shown in Figure 5.31.

$$\begin{aligned} V &= 250 \text{ N} \\ Q &= A\tilde{y} = \left(\frac{\pi D^2}{8}\right) \left(\frac{2D}{3\pi}\right) = 6.667 \times 10^{-7} \text{ m}^3 \\ I &= 7.854 \times 10^{-9} \text{ m}^4 \\ b &= D = 0.02 \text{ m} \\ \tau &= -1.1 \text{ MPa} \end{aligned} \quad (5.72)$$

This shear stress acts perpendicular and parallel to the neutral surface (so is τ_{xy}) at the neutral surface and does not vary along the length. Note that this stress is much smaller (an order of magnitude) than the bending stress. The choice to focus on another region of the structure is therefore validated as the transverse shear stress is not a concern.

With the stresses from all surrogate models fully-defined, we proceed to superposition. Begin by drawing a single stress element representing point A, as shown in Figure 5.32. This stress element lies on the top surface of the structure, at $(x, y, z) = (0, 0.01, 0)$. Looking at the element from above, x is directed to the right, y out of the page, and z downward.

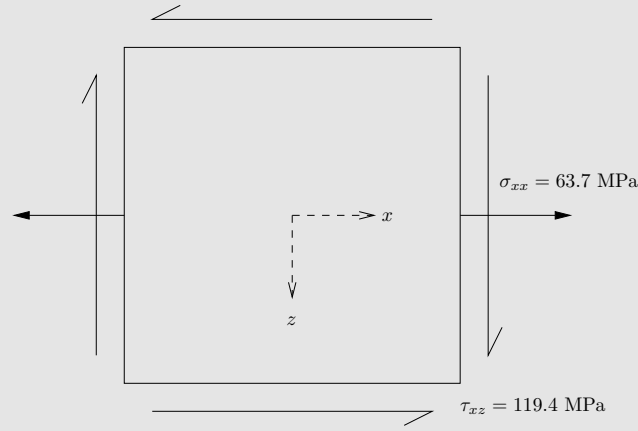


Figure 5.32: The stress element at point A

The stresses acting on this element include the normal stress from bending, σ_{xx} , and the shear stress from torsion. At this location, the shear stress is parallel to the xz -plane, so becomes τ_{xz} . As the two surrogate models do not result in any similar stresses, nothing must be added after it is included in the element. Thus, the total state of plane stress at point A is shown in Figure 5.32.

In this case, there are shear stresses present, so in-plane principal stresses must be calculated.

$$\begin{aligned}\sigma_{1p} &= \frac{63.7 + 0}{2} + \sqrt{\left(\frac{63.7 - 0}{2}\right)^2 + (119.4)^2} = 155.4 \text{ MPa} \\ \sigma_{2p} &= \frac{63.7 + 0}{2} - \sqrt{\left(\frac{63.7 - 0}{2}\right)^2 + (119.4)^2} = -91.7 \text{ MPa}\end{aligned}\tag{5.73}$$

Thus the triaxial principal stresses and maximum shear stresses are:

$$\begin{aligned}\sigma_1 &= 155.4 \text{ MPa} \\ \sigma_2 &= 0 \text{ MPa} \\ \sigma_3 &= -91.7 \text{ MPa} \\ \tau_{max} &= 123.5 \text{ MPa}\end{aligned}\tag{5.74}$$

5.6.2 Deformation under Combined Loading

Determination of total deformation under combined loading follows the same process, but can be somewhat more complicated. Due to the geometry of some structures, even small deformation can lead to perhaps unexpected displace-

ments of some points. Consider a simple bracket once more, shown in Figure 5.33(a). It has been determined that the upright segment acts as a beam, while the horizontal segment acts as both a beam and a bar. If we are to determine the displacement of the endpoint of the upright spar, the deformation of both segments in all modes must be detailed. First, the bar action of the horizontal spar moves the entire upright segment to the right (Figure 5.33(b)). The horizontal segment also bends downward, displacing the upright in the negative y -direction. However, as shown in Figure 5.33(c), the horizontal segment also becomes curved, with a slope of θ at point A. This rotates the entire upright about point A. Using the known slope, the location of point A, and the length of the upright, we can determine the new location of the endpoint B. It has displaced further to the right and slightly downward. Finally, the upright segment itself bends as a beam *perpendicular to its axis* (Figure 5.33(d)), further displacing the endpoint downward and to the right.

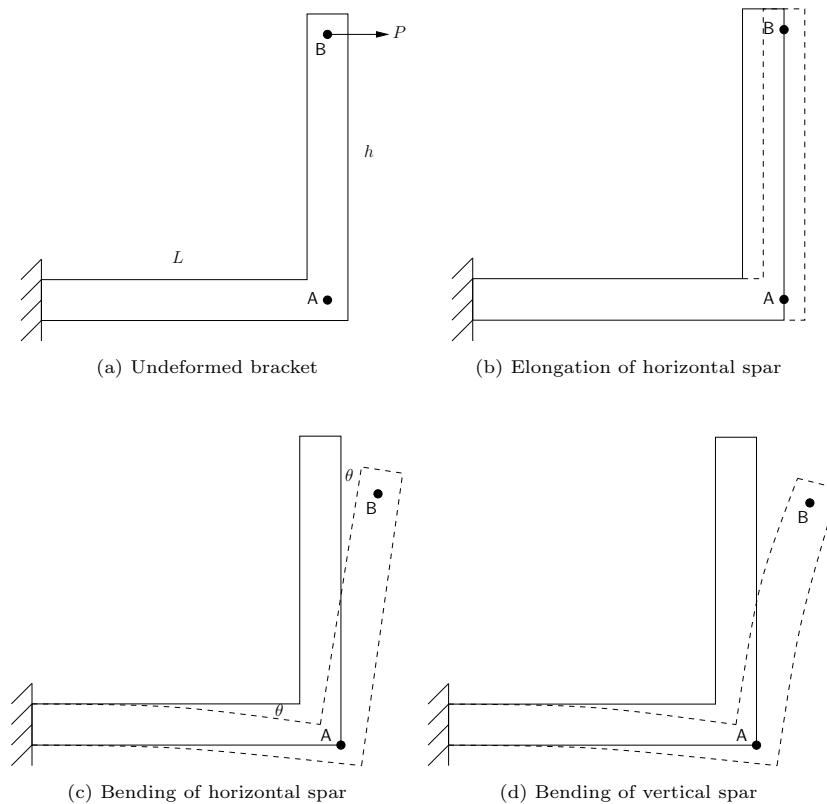


Figure 5.33: Deformation under combined loading

Fortunately, deformation is typically not a major factor in combined loading problems, with the focus generally on stress and material failure.

5.6.3 Discussion

- As discussed previously, care should be taken while completing combined loading problems that all assumptions and restrictions are satisfied. Small deformation, linearity, and the various surrogate model assumptions are all critical to the success of this method.
- This method is not without its flaws. Stress calculations can be quite inaccurate at geometric deconstruction points, particularly where the geometry of the total structure changes rapidly and induce significant stress concentrations. At no point do any of surrogate models provide a means for determining the stress in an elbow; we simply neglect it and consider either side, even though the elbow itself may be the greatest concern. Care should be taken that Saint-Venant's principle is applied appropriately so results can be interpreted correctly.
- Each of the surrogate models are an example of plane stress. Their combinations can therefore be plane stress as well, but analysts should be careful when drawing stress elements to be sure this is the case. The plane stress transformations are incorrect when used outside of plane stress problems.
- In modern engineering, it is not especially common to solve complicated combined loading problems by hand. It is still done in simpler cases, where the work may take less than an hour or two, but most more complex problems are solved using computational tools such as the finite element method (FEM). That being said, this is where the material covered in this section truly shows its value. Finite element simulations are easily constructed for three-dimensional problems, but can be very costly to run. Even projects of only middling complexity can take a powerful computer dozens or hundreds of hours to complete a full 3D analysis; each additional dimension essentially adds an exponent to the degrees of freedom in the numerical model. For example, if a 1D model has 10^6 degrees of freedom, the 3D equivalent could have more than $(10^6)^3 = 10^{18}$. However, with a clear understanding of the surrogate models, their necessary assumptions, and when and how to apply them, it is possible to significantly reduce the cost of simulations through proper use of bar, beam, etc. elements. Even in the modern age of technology, these fundamentals are key to effective and efficient problem solving.

5.7 Plane Strain*

Plane strain is similar in concept to plane stress. The out-of-plane components of the strain are all zero. Assuming we are working in the xy -plane, this

requires:

$$\begin{aligned}\varepsilon_{zz} &= 0 \\ \gamma_{xz} &= 0 \\ \gamma_{yz} &= 0\end{aligned}\tag{5.75}$$

Note that if ε_{xx} and ε_{yy} are nonzero, then Poisson effects in general would lead to a nonzero ε_{zz} . In other words, there must be a force in the z -direction opposing the Poisson effects. This z -force naturally gives rise to a stress σ_{zz} . That is, *plane strain is not equivalent to plane stress!*

Practical applications of plane strain are somewhat less common than plane stress, although there are some relevant examples. It is a reasonable representation of a structure that is very large in one dimension relative to the others; the strain along the large dimension, while not truly zero, is negligible compared to the other two, and the loading on the structure is primarily in the plane. The most common example is likely a reservoir dam, which is much longer than it is wide, constrained on either end, and loaded along its length normal to the plane by the pressure of the reservoir. Manufacturing processes such as rolling and drawing may fit well in this model as well because deformation in a given direction is constrained by the machinery.

Through appropriate use of geometry, similar to that shown in Figure 1.37, it is possible to derive coordinate transformations for plane strain:

$$\varepsilon_{x1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta)\tag{5.76}$$

$$\frac{\gamma_{xy1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin(2\theta) + \frac{\gamma_{xy}}{2} \cos(2\theta)\tag{5.77}$$

Note that the plane strain transformation equations (5.76) and (5.77) are analogous in form and function to the plane stress transformation equations (5.14), even though they represent different physics and were derived wholly independently of one another. Carrying this further, it is straightforward to develop expressions for in-plane principal and maximum shear strains.

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}\tag{5.78}$$

$$\frac{\gamma_{max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}\tag{5.79}$$

The plane strain transformations (5.76) and (5.77) were developed based on geometry, with no consideration of any other features. The plane stress transformations (5.14) were derived from force equilibrium. Although the two look similar, it stands to reason that this is coincidental and they are unrelated.

From the generalized Hooke's law, each component of shear stress depends only upon the corresponding component of shear strain. Consider the out-of-plane shear stresses and strains. In plane stress, τ_{xz} and τ_{yz} are zero, implying

that γ_{xz} and γ_{yz} are also zero. The same holds true in plane strain. Further recall from equation (5.11) and the discussion on triaxial stresses in Section 5.3.1 that σ_{zz} (which is zero in the case of plane stress) does not appear in the xy -force equilibrium equation. In the case of plane strain, σ_{zz} is nonzero. By this argument, though, that would not affect the plane stress transformation equations for the xy -plane. In other words, the plane stress transformations (5.14), and consequently the in-plane principal and maximum stress equations (5.22) and (5.24) hold for plane strain problems!

Likewise, the plane strain transformations (5.76) and (5.77) as well as principal and maximum shear strain equations (5.78) and (5.79) hold for plane stress. In both cases, we can freely transform stresses and strains. Accordingly, we can show that the principal strains occur in the same configuration as the principal stresses.

5.7.1 Strain Gauges

Although useful in modeling some physical situations, the true strength of the plane strain model is not in its representation of general applications but its use in generalizing measurement. It is possible to directly measure strain through the use of **strain gauges**. Strain gauges often consist of a conductive material whose electrical resistance changes with relative deformation; a variation of output voltage in a circuit containing the strain gage is directly proportional to the strain it experiences. Strain gauges may be used during experimentation, for detecting damage in buildings, for monitoring biomechanical motion, and more.

An individual strain gauge measures strain along its own axis. In order to measure strain in multiple directions, multiple strain gauges are required. These gauges are typically arranged in a **rosette**, a collection of three gauges at a single point oriented in different directions. One common example is the 45° rosette shown in Figure 5.34.

This rosette consists of three gauges. Gauges A and B are aligned with the x - and y - axes, while gauge C lies at 45° to both of them. Each gauge directly measures the normal strain in its direction. The strain transformations allow us to use this information to determine the shear strain γ_{xy} as well as strain in any other configuration, including principal and maximum shear strains.

Gauges A and B as measure the x -strain, $\varepsilon_x = \varepsilon_A$, and the y -strain, $\varepsilon_y = \varepsilon_B$. If this configuration is rotated 45° , then the normal strain would be measured by gauge C; in other words, $\varepsilon_{x1} = \varepsilon_C$. Applying this to the transformation equation (5.76), we can solve for γ_{xy} .

$$\begin{aligned}\varepsilon_C &= \frac{\varepsilon_A + \varepsilon_B}{2} + \frac{\varepsilon_A - \varepsilon_B}{2} \cos(2(45^\circ)) + \frac{\gamma_{xy}}{2} \sin(2(45^\circ)) \\ \implies \gamma_{xy} &= 2\varepsilon_C - \varepsilon_A - \varepsilon_B\end{aligned}\quad (5.80)$$

With the shear strain determined, strain in any configuration may be calculated through direct application of the transformation equations.

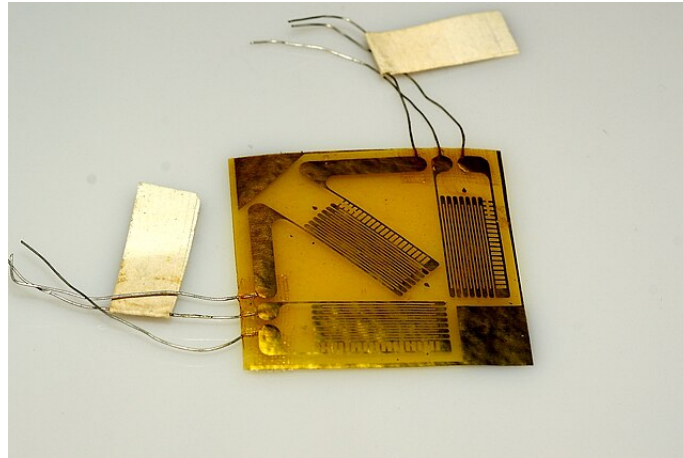


Figure 5.34: 45° strain gauge rosette ([Wikimedia Commons](#), [CCA 4.0](#))

5.8 Chapter Summary

Plane stress is a class of surrogate models containing bars, shafts, beams, and pressure vessels. The only requirement for a problem to reduce to plane stress is that out-of-plane stresses, both normal and shear, are negligible. Of course, for any of the subordinate 1D models to be used, their respective assumptions and restrictions must hold. By applying this restriction to the generalized Hooke's law, linear elasticity for plane stress can be defined.

Plane stresses may undergo orthogonal rotations of coordinates to reveal alternative stress components. These stress transformations are defined using only force equilibrium so are independent of material, linearity, or geometry; again, the only requirement is that out-of-plane stresses be zero. The configuration in which normal stresses reach their greatest magnitudes is called the principal configuration, and the stresses are the principal stresses. In the principal configuration, shear stress is zero. Oriented 45° from the principal configuration is that in which the shear stress is maximized. Normal stresses are not required to be zero in the maximum shear configuration.

In plane stress, the out-of-plane stresses are zero, but may still be considered. A material point is said to be in a state of triaxial stress when it is subjected only to three normal stresses. Since triaxial stress lacks shear components of the stress, it is a state of principal stress. In the case of plane stress, the third normal stress has a value of zero. This allows for relatively easy rotation of the coordinate system 45° out-of-plane to determine maximum shear stresses in other directions as well. Principal and maximum shear stresses have value as a means of predicting material failure.

Bars, shafts, and beams are each special cases of plane stress. A fourth class of surrogates, thin-walled pressure vessels, are also plane stress models. The model for pressure vessels has the following requirements:

1. The material is continuous, homogeneous, isotropic, linearly elastic, and quasistatic
2. The vessel is fully-enclosed with no openings
3. The structure contains a fluid pressurized above external ambient pressure
4. The pressure within the structure is uniform
 - The pressure of the fluid and the orientation and size of the tank are such that hydrostatic pressure due to weight of the fluid is negligible
5. The pressure vessel is cylindrical or spherical
6. The tank is thin-walled ($t \ll R$)
7. The tube is prismatic

If these conditions are satisfied, radial stress and shear stresses are negligible and the tank reduces to a state of plane stress in which the stresses are parallel to the surface of the vessel and depend upon pressure and tank dimensions.

It is noted that plane stress is defined at a material point, not necessarily for the entire structure. This is beneficial for combined loading problems. When deformation is small and all relevant relations are linear, the bar, shaft, beam, and pressure vessel models may be applied to a single structure and the results superpositioned, even if the geometry or loading results in a three-dimensional problem. All stresses at a given location are applied to a single stress element, with appropriate components summed, and transformations may be performed to determine principal and maximum shear stresses.

Plane strain is another reduction of three-dimensional mechanics analogous to plane stress. In this case all out-of-plane strains are zero. Plane stress and plane strain are mutually exclusive; despite their superficial similarity they almost never apply simultaneously due to Poisson effects. Plane strain can also undergo coordinate transformations. The plane strain transformations are derived geometrically, but are equivalent in form to the plane stress transformations. Careful observation shows that plane strain transformations can be applied to strains in a plane stress model and vice versa. The most ubiquitous use of plane strain is in the strain gauge rosette, a tool for direct strain measurement.

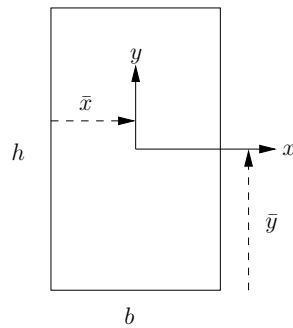
5.8.1 List of Symbols

σ :	Normal stress
τ :	shear stress
ε :	Normal strain
γ :	Shear strain
E :	Young's modulus
ν :	Poisson's ratio
σ_{x1} :	Stress in the $x1$ -coordinate system
σ_{x2} :	Stress in the $x2$ -coordinate system
θ :	Angle of rotation from the original coordinate system to the transformed system
θ_p :	Principal angle
σ_{1p} :	First in-plane principal stress
σ_{2p} :	Second in-plane principal stress
$\tau_{max,p}$:	Maximum in-plane shear stress
σ_1 :	First triaxial principal stress
σ_2 :	Second triaxial principal stress
σ_3 :	Third triaxial principal stress
τ_{max} :	Maximum shear stress
p :	Fluid pressure
R :	Internal radius of a pressure vessel
t :	Pressure vessel wall thickness
σ_H :	Hoop or circumferential stress
σ_A :	Axial or longitudinal stress
σ_M :	Membrane stress

Appendix A

Planar Geometry

A.1 Rectangle



$$A = bh$$

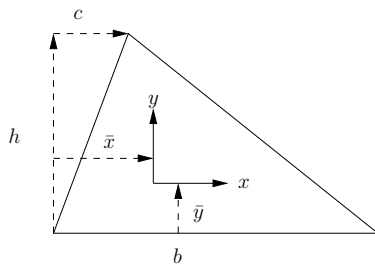
$$\bar{x} = \frac{b}{2}$$

$$\bar{y} = \frac{h}{2}$$

$$I_{xx} = \frac{bh^3}{12}$$

A.2 Triangle

Note that this is a generic scalene triangle. The expressions below hold for any triangular shape, but may be simplified to more familiar forms if the triangle is isosceles or right, for example.



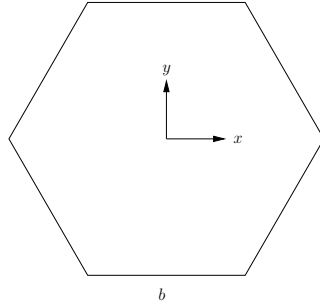
$$A = \frac{bh}{2}$$

$$\bar{x} = \frac{b+c}{3}$$

$$\bar{y} = \frac{h}{3}$$

$$I_{xx} = \frac{bh^3}{36}$$

A.3 Regular n -sided Polygon

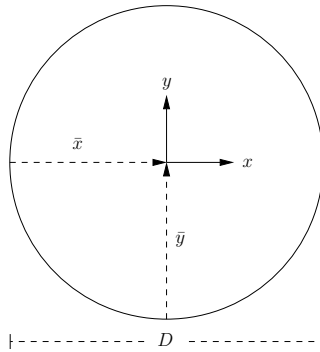


$$A = \frac{nb^2}{4 \tan\left(\frac{\pi}{n}\right)}$$

$$I_{xx} = \frac{nb^4}{192} \left(\frac{1}{\tan\left(\frac{\pi}{n}\right)} + \frac{3}{\tan^2\left(\frac{\pi}{n}\right)} \right)$$

\bar{x} , \bar{y} vary with n and orientation
but centered due to symmetry

A.4 Circle



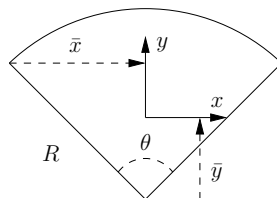
$$A = \frac{\pi D^2}{4}$$

$$\bar{x} = \frac{D}{2}$$

$$\bar{y} = \frac{D}{2}$$

$$I_{xx} = \frac{\pi D^4}{64}$$

A.5 Circular Sector



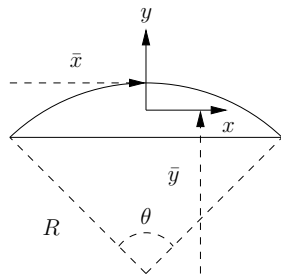
$$A = \frac{\theta}{2} R^2$$

$$\bar{x} = R \sin\left(\frac{\theta}{2}\right)$$

$$\bar{y} = \frac{4R}{3\theta} \sin\left(\frac{\theta}{2}\right)$$

$$I_{xx} = \frac{R^4}{8} (\theta + \sin(\theta))$$

A.6 Circular Segment



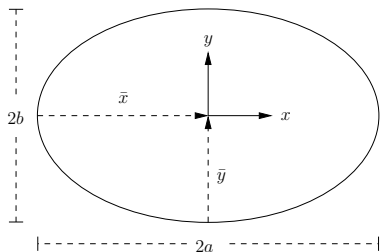
$$A = \frac{R^2}{2} (\theta - \sin(\theta))$$

$$\bar{x} = R \sin\left(\frac{\theta}{2}\right)$$

$$\bar{y} = \frac{4R}{3} \left(\frac{\sin^3\left(\frac{\theta}{2}\right)}{\theta - \sin(\theta)} \right)$$

$$I_{xx} = \frac{R^4}{8} \left(\theta - \sin(\theta) + 4 \sin^3\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \right)$$

A.7 Ellipse



$$A = \pi ab$$

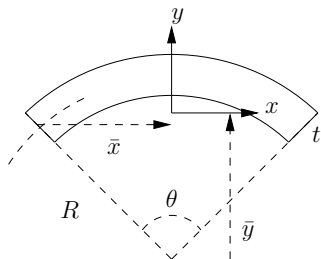
$$\bar{x} = a$$

$$\bar{y} = b$$

$$I_{xx} = \frac{\pi ab^3}{4}$$

A.8 Thin Circular Arc

Note: The thickness t must be much less than the midplane radius of the arc R .



$$A = \theta R t$$

$$\bar{x} = R \sin\left(\frac{\theta}{2}\right)$$

$$\bar{y} = \frac{2R \sin\left(\frac{\theta}{2}\right)}{\theta}$$

$$I_{xx} = \frac{R^3 t}{2} (\theta + \sin(\theta))$$

Appendix B

Material Properties

The following table lists properties for various materials. Care should be taken when using the table for real applications. The properties listed are generalized. For example, steel is listed as a material, but in truth steel is not a single material. There are low-carbon, high-carbon, stainless, and tool categories of steels, each with many different specific materials with their own unique properties. Furthermore, material processing can profoundly affect properties. Hot working, cold working, forging, tempering, case hardening, etc. all result in different final properties of the material.

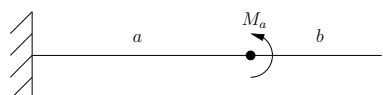
The values listed in this appendix are “average” properties in a sense. They are appropriate for any assignments associated with this text and may be used as a reasonable starting estimate for real applications, but more precise values should be found for specific materials.

Material	ρ (kg/m^3)	E (GPa)	G (GPa)	ν	α ($10^{-6}/\text{K}$)	S_y (MPa)	S_U (MPa)	Duct. (% EL)
Steel	7850	200	77	0.3	13	350	500	20
Aluminum	2750	70	26	0.33	23	300	450	15
Brass	8500	100	35	0.34	20	300	450	20
Rubber	1000	0.002	0.001	0.48	150	35	40	300
ABS	1000	2	1	0.38	120		45	100
Wood	150	10	4	0.4	4	100	200	10

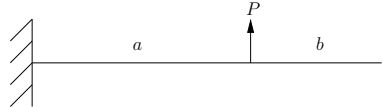
Appendix C

Beam Tables

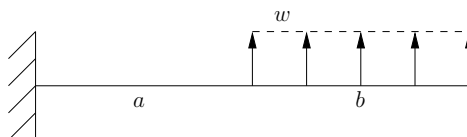
C.1 Cantilevered Beam with Couple


$$\theta(x) = \begin{cases} \frac{M_a x}{EI} & x \leq a \\ \frac{M_a a}{EI} & x \geq a \end{cases}$$
$$v(x) = \begin{cases} \frac{M_a x^2}{2EI} & x \leq a \\ \frac{M_a a}{2EI}(2x - a) & x \geq a \end{cases}$$

C.2 Cantilevered Beam with Point Force


$$\theta(x) = \begin{cases} \frac{Px}{2EI}(2a - x) & x \leq a \\ \frac{Pa^2}{2EI} & x \geq a \end{cases}$$
$$v(x) = \begin{cases} \frac{Px^2}{6EI}(3a - x) & x \leq a \\ \frac{Pa^2}{6EI}(3x - a) & x \geq a \end{cases}$$

C.3 Cantilevered Beam with Uniform Distributed Load



$$\theta(x) = \begin{cases} \frac{wx(L-a)}{2EI}(L+a-x) & x \leq a \\ \frac{w}{6EI}(x^3 - 3Lx^2 + 3L^2x - a^3) & x \geq a \end{cases}$$

$$v(x) = \begin{cases} \frac{wx^2(L-a)}{12EI}(3L+3a-2x) & x \leq a \\ \frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2 - 4a^3x + a^4) & x \geq a \end{cases}$$